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Linearization of Nonlinear Dynamical Systems: A Comparative Study

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Abstract

Linearization of nonlinear dynamical systems is a main approach in the designing and analyzing of such systems. Optimal linear model is an online linearization technique for finding a local model that is linear in both the state and the control terms. In this paper, a comparison between the performance of both optimal linear model and Jacobian linearization technique is conducted. The performance of these two linearization methods are illustrated using two benchmark nonlinear systems, these are inverted pendulum system; and Duffing chaos system. These two systems where chosen because they are inherently nonlinear unstable systems.

Keywords: Nonlinear systems; linearization; inverted pendulum.

1. Introduction

Linearization of nonlinear dynamical systems makes use of available literature in the linear system to design and analyze nonlinear systems [1]. Optimal linear model is a promising linearization technique that continues to find wide acceptance in the areas of nonlinear and chaos systems [2, 3, 4]. This method was first introduced by Teixeira and Zak [5].

In fact, typical approach to handle nonlinear systems is to utilize linearization at their operating points, including Jacobian analysis for local dynamics of control systems. Optimal linear model is another method that generates optimal local models; it is an online linearization technique for finding a local model that is linear in both the state and the control terms. This technique provides a new tool to control nonlinear systems and it is briefly described in the next section.

Furthermore, the inverted pendulum problem has been used as benchmark to motivate the study of nonlinear control systems and techniques [6, 7]. This system is inherently nonlinear unstable non-minimum phase system and provides a challenging system to test different control techniques [8].

In addition, Duffing system is well-known chaos attractor and has used to address many practical applications in Engineering and Science [9, 10]. In this paper two Duffing systems are synchronized together. Where synchronization is when two systems come to behave in accordance with each other as time passes.

The rest of paper is organized as follows. In Section 2, the optimal linear model is discussed and generation of linearized models around operating points is shown In Section 3, the effectiveness of the optimal linear model is demonstrated and its performance is compared with the Jacobian method performance. Finally, conclusions are presented in section 4.

2. Optimal Linear Model of Non-linear Systems

Consider a nonlinear system model

$$\dot{x}(t) = F(x(t)) + G(x(t))u(t), \tag{1}$$

where $F:\Re^n \rightarrow \Re^n$ and $G:\Re^n \rightarrow \Re^{nxm}$ are nonlinear function, $x(t)\in\Re^n$ is the state vector, and $u(t)\in\Re^m$ is the control input. The optimal linear model objective is to find linearized models of a nonlinear dynamical system around operating points and it is described as follows.

Suppose that it is desired to have a local linear model (A_{op}, B_{op}) at the *i-th* operation point of interest (x_{op}, u_{op}), which is not necessarily an equilibrium point of the system. Let the linearized model be given as

$$\dot{x}(t) = A_{op}x(t) + B_{op}u(t) \tag{2}$$

where A_{op} and B_{op} are constant matrices of appropriate dimensions. For this purpose, Taylor expansion method is commonly used in this case, however, the truncation used in this method results in an affined rather than linear model. Suppose that the operating point (x_{op}, u_{op}) is an equilibrium point, where $x_{op}(t)\in\Re^n$ and $u_{op}(t)\in\Re^m$, that is,

$$F(x_{op}) + G(x_{op})\,u_{op} = 0. \tag{3}$$

The linear model then is expressed as:

$$\begin{aligned}\frac{d}{dt}(x - x_{op}) &= F(x_{op}) + G(x_{op})u_{op} + A_{op}(x - x_{op}) + B_{op}(u - u_{op}) \\ &= A_{op}(x - x_{op}) + B_{op}(u - u_{op}).\end{aligned} \tag{4}$$

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The model (4) can be represented in the following form:

$$\dot{x}(t) = A_{op}x(t) + B_{op}u(t) - (A_{op}x_{op} + B_{op}u_{op}). \tag{5}$$

The state equation (5) is an affined rather than a linear model due to the non-vanishing constant term in (5). One exception is the trivial case where the equilibrium point is zero, which, however, cannot be ensured throughout a nonlinear control process. The goal is to construct a local model, linear in state and also linear in control, that can well approximate the dynamical behavior of (1) around the operating point (x_{op}, u_{op}). In other words, two constant matrices, A_{op} and B_{op}, are to be found such that they are located in a neighborhood of x_{op},

$$F(x) + G(x)u \approx A_{op}x + B_{op}u \tag{6}$$

for any u,

and

$$F(x_{op}) + G(x_{op})u = A_{op}x_{op} + B_{op}u \tag{7}$$

for any u.

Since the control input u is to be designed, it is arbitrary, and therefore

$$B_{op} = G(x_{op}), \tag{8}$$

Furthermore,

$$F(x) \approx A_{op}x \tag{9}$$

And

$$F(x_{op}) = A_{op}x_{op} \tag{10}$$

Now let a_{op}^T denotes the *i*-th row of the matrix A_{op}. Then

$$F_i(x) \approx a_i^T x, \qquad i = 1,2,3,\cdots,n \tag{11}$$

and

$$F_i(x_{op}) = a_{op}^T x_{op}, \; i = 1,2,3,\cdots,n \tag{12}$$

where $F_i : \Re^n \to \Re$ is the *i*-th component of F. Then, expanding the left-hand side of (1) about x_{op}, and neglecting the second and higher order terms, the following equation can be obtained

$$F_i(x_{op}) + [\nabla F_i(x_{op})]^T (x - x_{op}) \approx a_i^T x, \tag{13}$$

where $\nabla F_i(x_{op}) : \Re^n \to \Re^n$ is the gradient column vector of F_i evaluated at x_{op}. Now, using (12), equation (13) can be written as

$$[\nabla F_i(x_{op})]^T (x - x_{op}) \approx a_i^T (x - x_{op}), \tag{14}$$

in which x is arbitrary but should be close to x_{op} so that the approximation is good. To determine a constant vector, a_i^T, such that it is as close as possible to $[\nabla F_i(x_{op})]^T$ and satisfies $a_i^T x_{op} = F_i(x_{op})$, consider the following constrained minimization problem:

$$\min \; E := \tfrac{1}{2}\left\| \nabla F_i(x_{op}) - a_i \right\|_2^2$$

subject to

$$a_i^T x_{op} = F_i(x_{op}) \tag{15}$$

Given that this is a constrained optimization problem; therefore, the first order necessary condition for a minimum of E is also sufficient, which is

$$\nabla_{a_i} E + \lambda \nabla_{a_i} (a_i^T x_{op} - F_i(x_{op})) = 0 \tag{16}$$

$$a_i^T x_{op} = F_i(x_{op}), \tag{17}$$

where λ is the Lagrange multiplier and the subscript a_i in ∇_{a_i} indicates that the gradient is taken with respect to a_i. Then

$$a_i - \nabla F_i(x_{op}) + \lambda x_{op} = 0. \tag{18}$$

Recalling the case where $x_{op} \neq 0$, so by solving (18), the following can be obtained

$$\lambda = \frac{x_{op}^T \nabla F_i(x_{op}) - F_i(x_{op})}{\left\| x_{op} \right\|_2^2}. \tag{19}$$

Substituting this λ into (18) gives

$$a_j = \nabla F_i(x_{op}) + \frac{F_i(x_{op}) - x_{op}^T \nabla F_i(x_{op})}{\left\| x_{op} \right\|_2^2} x_{op} \tag{20}$$

where $x_{op} \neq 0$ It is easily verified that when $x_{op} = 0$ equation (18) yields

$$a_i = \nabla F_i(x_{op}) \tag{21}$$

which is also a special case of (20).

3. Simulation Examples

Most often, control algorithms are tested on standard nonlinear models, and the objective is to first find the linearized model then to design suitable controller based on our skills with linear systems. These linearized models of the nonlinear model are valid only for small deviations of the state values from their nominal value. Such a nominal value is called the equilibrium point. Therefore, the linear models are acceptable around a small range of the operating point. In this section, the effectiveness of the above method described in previous section will be presented and its performance is compared with Jacobian method performance.

3.1. Inverted Pendulum:

The inverted pendulum, shown in Figure 1, is highly nonlinear system which can be considered as an important benchmark system for controller testing.

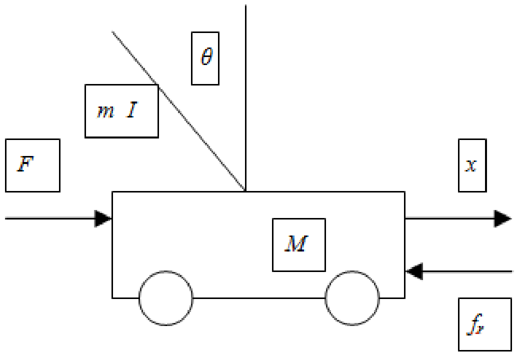

Figure 1: Inverted pendulum system.

The nonlinear dynamical equations of motion is given by [11].

$$(m+M)\ddot{x} + f_r\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F$$
$$(I+ml^2)\ddot{\theta} - mgl\sin\theta + ml\ddot{x}\cos\theta = 0 \qquad (22)$$

where m is the pole mass, M is the cart mass, f_r is the cart friction coefficient, x is the horizontal displacement, l is the pole length, θ is the angle of the pole from upright position, F is the applied force on the cart, I is the pole moment of inertia, and g is the gravity. The state space model can be presented as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-f_r(I+ml^2)}{(m+M)I+mMl^2} & \frac{m^2l^2g\cos x_3}{(m+M)I+mMl^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{mlf_r\cos x_3}{mI+M(I+ml^2)} & \frac{-(m+M)mgl}{mI+M(I+ml^2)} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{(I+ml^2)}{(m+M)I+mMl^2} \\ 0 \\ \frac{-ml\cos x_3}{mI+M(I+ml^2)} \end{bmatrix} u \qquad (21)$$

Where

$[x_1\ x_2\ x_3\ x_4]^T = [\dot{x}\ \ddot{x}\ \dot{\theta}\ \ddot{\theta}]^T$.

Given equations (20) and (21) the optimal model

$$A_{op} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a & b(\cos x_3) - b(\sin x_3)x_3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & d(\cos x_3) & -d(\sin x_3)x_2 + e & 0 \end{bmatrix} \qquad (24)$$

$$B_{op} = \begin{bmatrix} 0 \\ c \\ 0 \\ h\cos x_3 \end{bmatrix} \quad for \quad \|x\|_2^2 = 0\cos x_3$$

And

$$A_{op} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a + \frac{b(\sin x_3)x_3x_2}{\|x\|_2^2} & b(\cos x_3) - b(\sin x_3)x_3 + \frac{b(\sin x_3)x_3^2}{\|x\|_2^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & d(\cos x_3) + \frac{d(\sin x_3)x_3^2}{\|x\|_2^2} & -d(\sin x_3)x_2 + e + \frac{d(\sin x_3)x_3x_2}{\|x\|_2^2} & 0 \end{bmatrix} \qquad (25)$$

$$B_{op} = \begin{bmatrix} 0 \\ c \\ 0 \\ h\cos x_3 \end{bmatrix} \quad for \quad \|x\|_2^2 \neq 0$$

where

$$\|x\|_2^2 = [x_1\ x_2\ x_3\ x_4][x_1\ x_2\ x_3\ x_4]^T,\ a = \frac{-f_r(I+ml^2)}{(m+M)I+mMl^2},\ b = \frac{m^2l^2g}{(m+M)I+mMl^2}$$
$$c = \frac{(I+ml^2)}{(m+M)I+mMl^2},\ d = \frac{mlf_r}{mI+M(I+ml^2)},\ e = \frac{-(m+M)mgl}{mI+M(I+ml^2)}$$
$$h = \frac{-ml}{mI+M(I+ml^2)}$$

The Jacobian model is gradient of the system is given by:

$$\frac{\partial F_i(x)}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a & b\cos x_3 - b(\sin x_3)x_3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & d\cos x_3 & -d(\sin x_3)x_2 + e & 0 \end{bmatrix} \qquad (26)$$

The inverted pendulum parameters are assigned the following values [11], m = 0.23 kg, M = 2.4 kg, l = 0.38 m. fr = 0.05 Ns/m, I = 0.099 kg.m2, and g = 9.81 m/s2. Furthermore, the simulation for the controllers, with pole placement at poles -1±j1 and -2±j1, was carried out in SimuLink of MatLab using a fifth-order Dormand-Prince algorithm with a fixed integration step of 0.005 and initial condition of [0, 0, 0, 1]T. The performances for both optimal linear model and Jacobian method are displayed in Figure 2. It is obvious that first method has less overshoot and better performance than the second method.

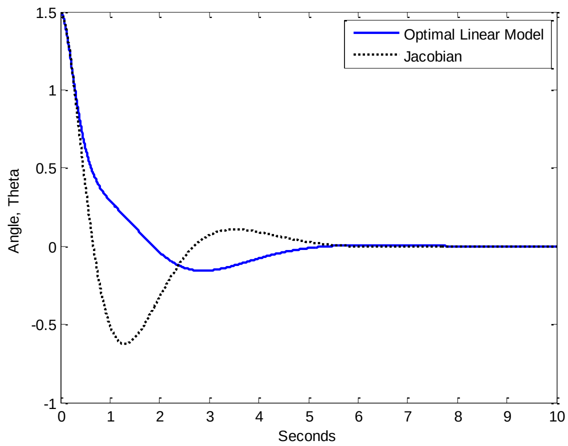

Figure 2: Comparison between Optimal linear model and Jacobian method.

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3.2. Synchronization of Duffing System:

The chaotic Duffing system is a popular benchmark example in the study of nonlinear system. The Duffing system can be expressed as [4, 12]:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -p_1 x_2 - p_2 x_1 - p_3 x_1^3 + q\cos(wt) \end{aligned} \qquad (27)$$

with parameters p1 = -1.1, p2 = 0.4, p3 = 1, q = 1.8 and w =1.8, the system is chaotic and the attractor for uncontrolled system is shown Figure 3 below.

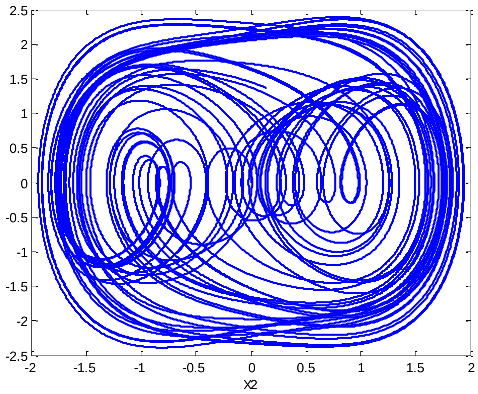

Figure 3: Duffing system chaotic attractor.

Applying equations (20) and (21), the optimal linear model is derived as:

$$A_{op} = \begin{bmatrix} 0 & 1 \\ -p_2 - 3p_3 x_{k1}^2 & -p_1 \end{bmatrix} \qquad for \quad \|x\|_2^2 = 0$$

$$A_{op} = \begin{bmatrix} 0 & 1 \\ -p_2 - 3p_3 x_1^2 + \frac{2p_3 x_1^4}{\|x\|_2^2} & -p_1 + \frac{2p_3 x_1^3 x_2}{\|x\|_2^2} \end{bmatrix} \quad for \quad \|x\|_2^2 \neq 0 \qquad (28)$$

$$B_k = [0 \quad 1]^T$$

and

$$C_k = [1 \quad 0]$$

where and $\|x\|_2^2 = [x_1, x_2]^T [x_1, x_2]$ is the square magnitude of the operating point.

In general, synchronization is when two systems come to behave in accordance with each other as time passes; an example is the transmitter / receiver unit in communication system. The goal here is to consider the synchronization problem from the point of view of control theory [3]. Mathematically speaking, consider two dynamical systems,

$$\begin{aligned} \dot{x}(t) &= f(x(t)), \quad x(0) = x_0 \\ y(t) &= h(x(t)) \end{aligned} \qquad (29)$$

and

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{f}(\hat{x}(t), y(t)), \quad \hat{x}(0) = \hat{x}_0 \\ \hat{y}(t) &= h(\hat{x}(t)) \end{aligned} \qquad (30)$$

the two systems are said to be synchronized if

$$\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0 \qquad (31)$$

For the optimal linearized model

$$\begin{aligned} \dot{x}(t) &= A_{op} x(t) + B_{op} u(t), \; x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \qquad (32)$$

with A_{op} and B_{op} are the matrices from optimal linear model, then an observer can be designed as

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + K_o(y(t) - \hat{y}(t)), \; \hat{x}(0) = \hat{x}_0 \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned} \qquad (33)$$

where K_o is the observer gain matrix.

The simulation for this synchronization was carried out in SimuLink of MatLab using a fifth-order Dormand-Prince algorithm with a fixed integration step of 0.005 and initial condition of [0, 5]T. Note that state x2 is the state that not measurable. The performance of synchronization for Duffing system is shown in Figure 4 for the optimal linear model and in Figure 5 for the Jacobian method. It is obvious that first method is superior to the second method, and it was able to synchronize completely with non-measurable state x2 but the Jacobian method fail to do so and the error was huge for most of the time.

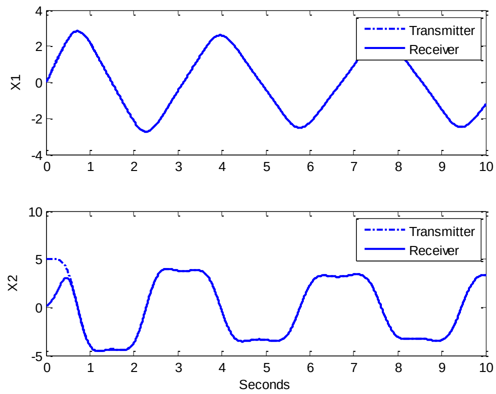

Figure 4: Synchronization of Duffing system using optimal linear model.

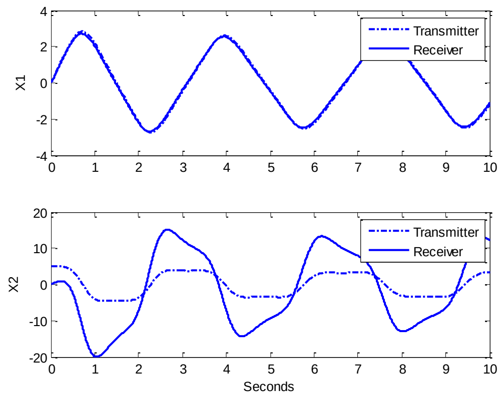

Figure 5: Synchronization of Duffing system using Jacobian method.

4. Conclusions

A linearization technique of optimal linear model was briefly presented in this paper, and its performance was compared with a popular Jacobian method. Two typical nonlinear benchmark examples were used to compare the two linearization methods; these are inverted pendulum and chaotic Duffing system. In the inverted pendulum example, the controllers were designed with pole placement for both cases, and as shown in Figure 2 the performance of first method was superior to the second method with much less overshoot. Furthermore, the synchronization of Duffing system was performed, as shown by the convincing simulation results in Figure 4 and 5, the optimal linear model was able to perfectly reproduce the non-measurable state x2, but the Jacobian failed to do so with large error. It is obvious from these results that optimal linear model has better performance than the Jacobian method, especially when the system is highly nonlinear. Future work might be in conducting performance comparison between optimal linear model and feedback linearization method.

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