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Conceptual Understanding of Mass and Stiffness Fixed Points of Discrete Vibrational Systems

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Abstract

The presence of mass and stiffness fixed points in the frequency responses of vibrational systems may greatly affect the design of these systems. In this paper, the physical reason for the occurrence of mass and stiffness fixed points and the relationship between them and the phenomenon of internal absorber are investigated. It is found that the frequencies at which mass and stiffness fixed points occur, represent eigenfrequencies of subsystems of the whole vibrational system. Furthermore, it is found that the mass and stiffness fixed points are strongly related with the phenomenon of internal absorbers.

Keywords: Mass Fixed Points, Stiffness Fixed Points, Vibration Control, Internal Absorber.

1. Introduction

The presence of damping, mass, and stiffness fixed points in the frequency responses of vibrating systems may complicate their vibration control since these fixed points can only be recognized if the parameters of the system are varied. A mass fixed point is an intersection of the frequency responses of a dynamic system for different values of the mass. Damping and stiffness fixed points are similarly defined. At a fixed point frequency, the vibration amplitude remains constant, independent of the values of the varied parameters. Thereto, when the operating frequency lie close to a mass (stiffness) fixed point frequency, then the amplitude of vibration cannot be effectively controlled by varying the values of masses (stiffnesses).

In addition to their dependence on the masses and stiffnesses of the dynamic system, the mass and stiffness fixed points are dependent on the location of the force application. That is because the location of the force application affects the phenomenon of the internal absorber, which is related with the force balance on different masses of the system.

Mass and stiffness fixed points may be used to design vibrational systems with zero or constant amplitudes for some masses of a system which can even include variable masses or stiffnesses.

Damping fixed points of systems with one and two degrees of freedom were treated in connection with vibration absorption and vibration isolation by many authors, including Den Hartog [1] and Klotter [2]. Bogy and Paslay [3] used the damping fixed points to obtain optimal damping for the purpose of minimizing the maximum steady state response of a particular linear damped two-degree-of-freedom vibratory system. Henney and Raney [4] used the damping fixed points to find approximate analytical expressions for optimum damping for a uniform beam forced and damped in four different configurations. Dayou [5] examined the fixed points theory for global vibration control of a continuous structure using vibration neutralizer.

Mass, stiffness, and damping fixed points of a system with two degrees of freedom were considered by Abu-Hilal [6], where the frequencies at which damping, mass, and/or stiffness fixed points occur and their amplitudes were determined analytically. Also Abu-Hilal [7] presented a procedure for determining the mass and stiffness fixed point frequencies of vibratory discrete linear system with n degrees of freedom. To verify the given procedure, all mass and stiffness fixed point frequencies of a system with three degrees of freedom were determined in closed forms.

In this paper the nature of the mass and stiffness fixed points of vibratory discrete linear dynamic systems and their physical meaning are investigated. Furthermore, the relationship between mass and stiffness fixed points and the phenomenon of internal absorber are studied. The vibration amplitudes at fixed points frequencies of an undamped system with three degrees of freedom as shown in Fig. 1 are determined and discussed. Although a three degrees of freedom system is studied in this contribution, the obtained results are general and applicable to systems

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with n degrees of freedom. A three degrees of freedom system is used in this study in order to obtain the fixed points frequencies and their amplitudes in closed forms.

2. Mathematical Formulation and Implementation

The equation of motion of an undamped linear system with n degrees of freedom is given as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \quad (1)$$

where **M**, **K**, and **x**, are the mass matrix, the stiffness matrix, and the displacement vector of the system, respectively, and **F** is the excitation force vector. If a harmonic force $\mathbf{F} = \mathbf{P}\cos\omega t$ is assumed, where $\mathbf{P} = [P_1, \ldots, P_n]^{\mathrm{T}}$ is the vector of the force amplitudes and ω is the circular excitation frequency, then the steady-state displacement vector of the system is obtained by using the solution

$$\mathbf{x} = \mathbf{X}\cos\omega t \quad (2)$$

Substituting Eq. (2) into Eq. (1) and simplifying yields

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{X} = \mathbf{P} \quad (3)$$

where $\mathbf{X} = [X_1, \ldots, X_n]^{\mathrm{T}}$ is the vector of the displacement amplitudes, $X_j = X_{j1} + \cdots + X_{jn}$ is the vibration amplitude of mass j, and X_{jg} ($j, g = 1,2,\ldots, n$) is the frequency response of mass j due to a force P_g applied at position g, with all other forces equal to zero (i.e.,

$P_1 1 = \cdots = P_1(g-1) = P_1(g+1) = \cdots = P_1 n = 0$).

In this contribution we set $P_g = P_0$.

The frequency response X_{jg} can be obtained from Eq. (3) and written in a bilinear form as given in [7] as:

$$X_{jg} = P_g \frac{a_1 e_i + a_2}{a_3 e_i + a_4} \quad (4)$$

where e_i represents a stiffness k_i or a mass m_i and a_1, a_2, a_3, and a_4 are polynomials in the variable ω^2.

The frequency response X_{jg} has a mass or a stiffness fixed point if [7]

$$\frac{a_1}{a_3} = \frac{a_2}{a_4} \quad (5)$$

The frequencies of these fixed points are determined by equating X_{jg} to two different values of e_i.

Using Eq. (5) or the procedure presented in [7], we obtain the frequencies of the mass and stiffness fixed points of the three mass system shown in Fig. 1. These frequencies are given in third row of Table 1, where

$$\omega_1 = 0 \quad (6)$$

$$\omega_2 = \sqrt{k_1/m_1} \quad (7)$$

$$\omega_3 = \sqrt{k_3/m_3} \quad (8)$$

$$\omega_4 = \sqrt{(k_1 + k_2)/m_1} \quad (9)$$

$$\omega_5 = \sqrt{k_3(m_2 + m_3)/m_2 m_3} \quad (10)$$

$$\omega_{6,7} = \sqrt{\frac{1}{2m_1m_2}[b_1 \mp \sqrt{b_1^2 - b_2}]} \quad (11)$$

$$\omega_{8,9} = \sqrt{\frac{1}{2m_1m_2}[b_3 \mp \sqrt{b_3^2 - b_4}]} \quad (12)$$

$$\omega_{10,11} = \sqrt{\frac{1}{2m_2m_3}[b_5 \mp \sqrt{b_5^2 - b_6}]} \quad (13)$$

$$b_1 = k_1m_2 + k_2(m_1 + m_2) \quad (14)$$

$$b_2 = 4k_1k_2m_1m_2 \quad (15)$$

$$b_3 = k_1m_2 + k_2(m_1 + m_2) + k_3m_1 \quad (16)$$

$$b_4 = 4(k_1k_2 + k_1k_3 + k_2k_3)m_1m_2 \quad (17)$$

$$b_5 = k_2m_3 + k_3(m_2 + m_3) \quad (18)$$

$$b_6 = 4k_2k_3m_2m_3 \quad (19)$$

Examining these frequencies yields that these frequencies are the natural frequencies of the systems shown in Fig. 2 which represent subsystems of the original system shown in Fig.1.

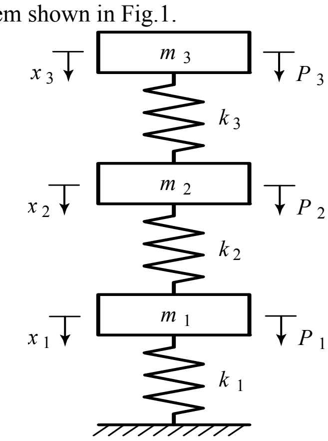

Fig. 1. Three-degree-of-freedom system.

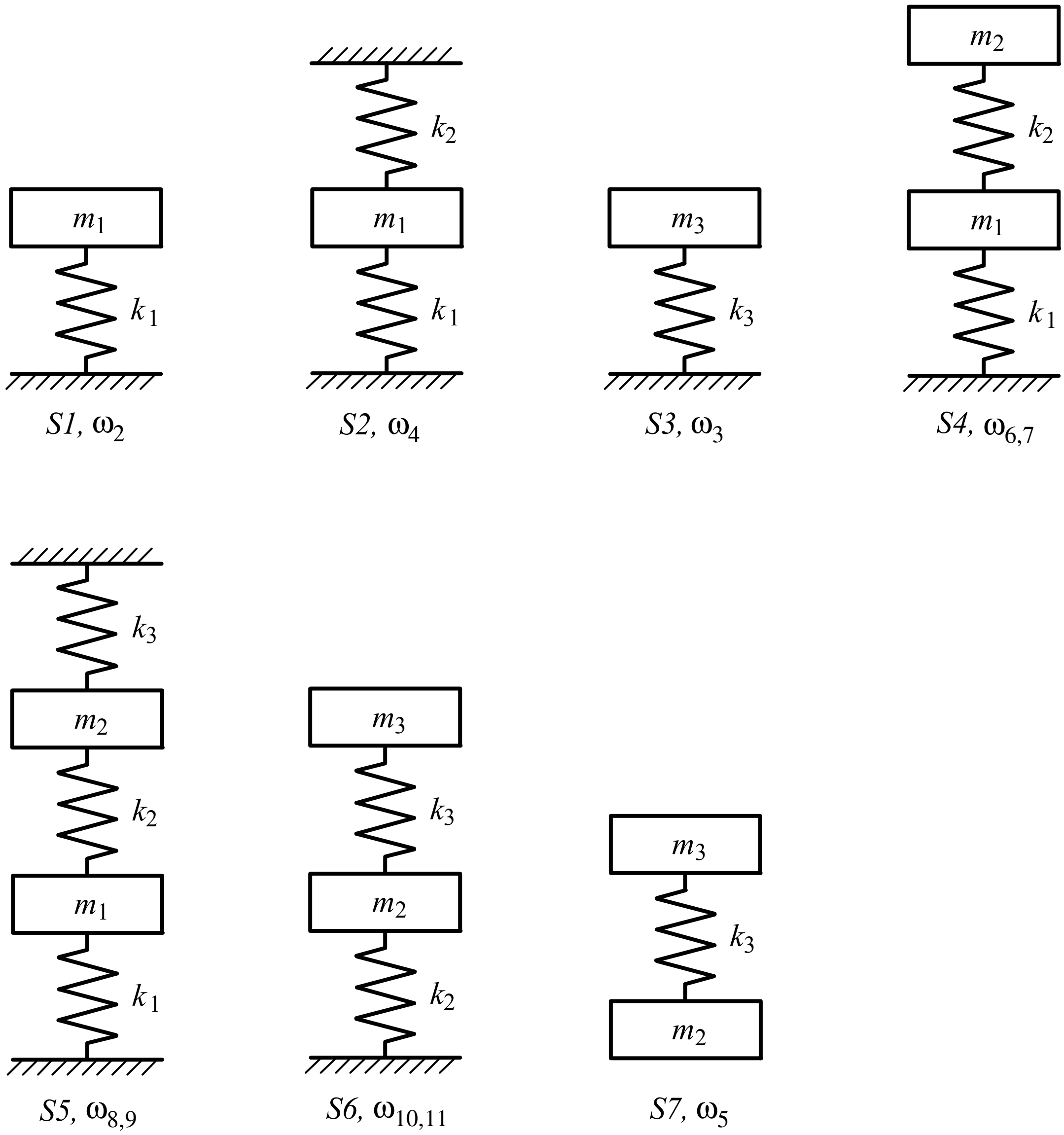

Fig. 2. Subsystems of a three-degree-of-freedom system.

In general, we can conclude, that the frequencies of mass and stiffness fixed points of vibrational linear discrete systems are natural frequencies of subsystems of the whole system.

The vibrational amplitudes A_{ij} at the fixed points frequencies of the system considered are listed in Table 1. Empty cells mean that the dynamic responses X_{ij} have no fixed points by varying the corresponding parameters k_i or m_i. For instance, from the fourth row of the table we can read that the dynamic response X_{11} has no stiffness fixed points by force application on mass m_1 with the frequency ω_3 and varying the stiffnesses k_1 or k_2. Other cells provide values of amplitudes A_{ij} as provided in the appendix.

3. Conceptual Analysis of Results

The following is conceptual discussion of the results given in Table 1.

3.1. Fixed points at the frequencies ω_{10} and ω_{11}

By a force application on mass m_1 in Fig.1 with an excitation frequency ω equal to one of the two natural frequencies ω_{10}, ω_{11} of the subsystem *S6* shown in Fig. 2, the subsystem *S6* serves in this case as an internal absorber to the subsystem *S1* shown in Fig.2. That is because at these frequencies the force transmitted from the spring k_2 to the mass m_1 is equal but opposite to the force acting there, so that m_1 remains at rest (x_1=0). The subsystem *S6* vibrates at ω_{10} in its first mode and at ω_{11} in its second mode with proportional amplitudes. In both cases, the amplitude A_{21} of mass m_2 is equal to P_0/k_2, because of the force balance at m_1 ($k_2A_{21}\cos(\omega_i t) = P_0\cos(\omega_i t)$, $_{i=10,11}$). The amplitudes of mass m_3 are then obtained from the eigenvectors of system *S6* and are given as

$$A_{31} = \frac{k_3}{k_3 - m_3\omega_i^2} A_{21}, \quad i = 10,11 \tag{20}$$

Table 1. Mass and stiffness point frequencies ω_i and their amplitudes A_{ij} for the three-degree-of-freedom-system shown in Fig.1

	Varying parameter																					
	k_1			k_2				k_3				m_1				m_2			m_3			
	ω_3	ω_{10}	ω_{11}	ω_1	ω_2	ω_3	ω_5	ω_1	ω_4	ω_6	ω_7	ω_1	ω_3	ω_{10}	ω_{11}	ω_1	ω_3	ω_4	ω_1	ω_4	ω_8	ω_9
A_{11}		0	0	C_1			C_{10}	C_1				C_1		0	0	C_1	C_8		C_1			
A_{21}	0	C_4	C_4	C_1	C_{12}	0	C_{10}	C_1	C_4			C_1	0	C_4	C_4	C_1	0	C_4	C_1	C_4		
A_{31}		C_{15}	C_{16}	C_1	C_{14}		C_{11}	C_1		C_{19}	C_{20}	C_1		C_{15}	C_{16}	C_1	C_9	C_7	C_1		C_{17}	C_{18}
A_{12}	0	C_4	C_4	C_1	C_{12}	0	C_{10}	C_1	C_4			C_1	0	C_4	C_4	C_1	0	C_4	C_1	C_4		
A_{22}	0				C_{12}	0		C_2	0			C_2	0			C_2	0	0	C_2	0		
A_{32}	C_5				C_{14}	C_5		C_2	0	C_{21}	C_{22}	C_2	C_5			C_2	C_5	0	C_2	0	C_5	C_5
A_{13}		C_{15}	C_{16}	C_1	C_{14}		C_{11}	C_1		C_{19}	C_{20}	C_1		C_{15}	C_{16}	C_1	C_9	C_7	C_1		C_{17}	C_{18}
A_{23}	C_5				C_{14}	C_5		C_2	0	C_{21}	C_{22}	C_2	C_5			C_2	C_5	0	C_2	0	C_5	C_5
A_{33}					C_{13}					C_{21}	C_{22}	C_3				C_3		C_6	C_3		0	0

All three frequency responses X_{i1} have mass and stiffness fixed points at the frequencies ω_{10} and ω_{11} since their amplitudes A_{i1} are independent of k_1 and m_1 at these frequencies. These frequency responses are shown in Fig. 3, first column for different values of m_1.

Also if the force P acts on the masses m_2 or m_3, then the frequency responses $X_{1i,\ i=2,3}$ stay having mass and stiffness fixed points at ω_{10} and ω_{11} (but with nonzero amplitudes), that is because of the symmetry of the system matrices **M** and **K** of the whole system shown in Fig. 1. On the other hand, the fixed points at ω_{10} and ω_{11} in the frequency responses of masses m_2 and m_3 vanish in this case. Figure 3, second column shows dynamic responses for different values of m_1 by force application on mass m_2.

3.2. Fixed points at the frequencies ω8 and ω9

By force application on mass m_3 in Fig.1 with an excitation frequency ω equal to one of the two natural frequencies ω_8 and ω_9 of the subsystem *S5* given in Fig. 2, this system serves as an internal absorber to the mass m_3. That is because at these frequencies the force transmitted from the spring k_3 to the mass m_3 is equal but opposite to the excitation force acting there, so that m_3 remains at rest ($x_3 = 0$). The subsystem *S5* vibrates at ω_8 in its first mode and at ω_9 in its second mode with proportional amplitudes. In both cases, the amplitude of mass m_2 is equal to P_0/k_3, because of the force balance at m_3 ($k_3x_2 = P$). The amplitude of mass m_1 is then obtained from the eigenvectors of system *S5* and given as

$$A_{13} = \frac{k_2}{k_1 + k_2 - m_1\omega_i^2} A_{23} \quad i = 8{,}9 \tag{21}$$

The frequency responses $X_{i3},\ _{i=1,2,3}$ have mass fixed points at the frequencies ω_8 and ω_9 since their amplitudes A_{i3} are independent of the values of m_3 at these frequencies.

Also if the force P acts on the masses m_1 or m_2, then mass m_3 stays keeping its fixed points at the frequencies ω_8 and ω_9, because of the symmetry of the mass and stiffness matrices of the whole system which leads to $X_{ij}=X_{ji},\ _{i,j=1,2,3}$. The frequency responses of masses m_1 and m_2 possess in this case no fixed points more.

3.3. Fixed points at the frequencies ω6 and ω7

By force application on mass m_3 in Fig. 1, the amplitudes of all three masses A_{i3} remain constant at the frequencies ω_6 and ω_7 independent of the values of k_3. This means that at these frequencies, the spring k_3 remains undeformed, so that at these frequencies $x_3 = x_2$, i.e. there is no relative motion between the masses m_2 and m_3. Hence by varying the values of k_3, the frequency responses X_{i3} get stiffness fixed points at ω_6 and ω_7 as shown in Fig.4.

By force application on the masses m_1 or m_2 at the frequencies ω_6 or ω_7 and varying the values of k_3, the stiffness fixed points of the dynamic responses X_{11}, X_{12}, X_{21}, and X_{22} vanish, where X_{31} and X_{32} stay having fixed points at ω_6 and ω_7 because of the symmetry of the mass and stiffness matrices of the whole system.

3.4. Fixed points at the frequencies ω2 and ω5

Varying the values of k_2 yields stiffness fixed points at the frequencies ω_2 and ω_5. At the frequency ω_2, all the frequency responses X_{ij}, except X_{11} have stiffness fixed points since their amplitudes A_{ij} at this frequency are independent of the stiffness k_2. At the frequency ω_5, the frequency responses X_{11}, X_{21}, X_{31}, X_{12}, and X_{13} have stiffness fixed points.

By force application on mass m_2 or m_3 at the frequency $\omega_5 = \sqrt{\frac{k_1}{m_1}}$, the natural frequency of the system *S1*, the system *S7* vibrates in its own way unaffected from the spring k_2, as if this spring does not exist.

In order to maintain the whole system connected by the force application on the masses m_2 or m_3 at the frequency ω_2, and at the same time the spring k_2 remains undeformed, the amplitude of mass m_1 must equal to the amplitude of the neighborhood mass m_2 in this case as shown in Table 1, 6th column; i.e. $X_{12}=X_{22}$, and $X_{13}=X_{23}$.

By force application on m_2, the vibration amplitudes of masses m_2 and m_3, respectively, become:
respectively, become:

$$A_{22} = \frac{m_1(k_3m_1 - k_1m_3)P_0}{k_1[k_1m_2m_3 - k_3m_1(m_2 + m_3)]} \tag{22}$$

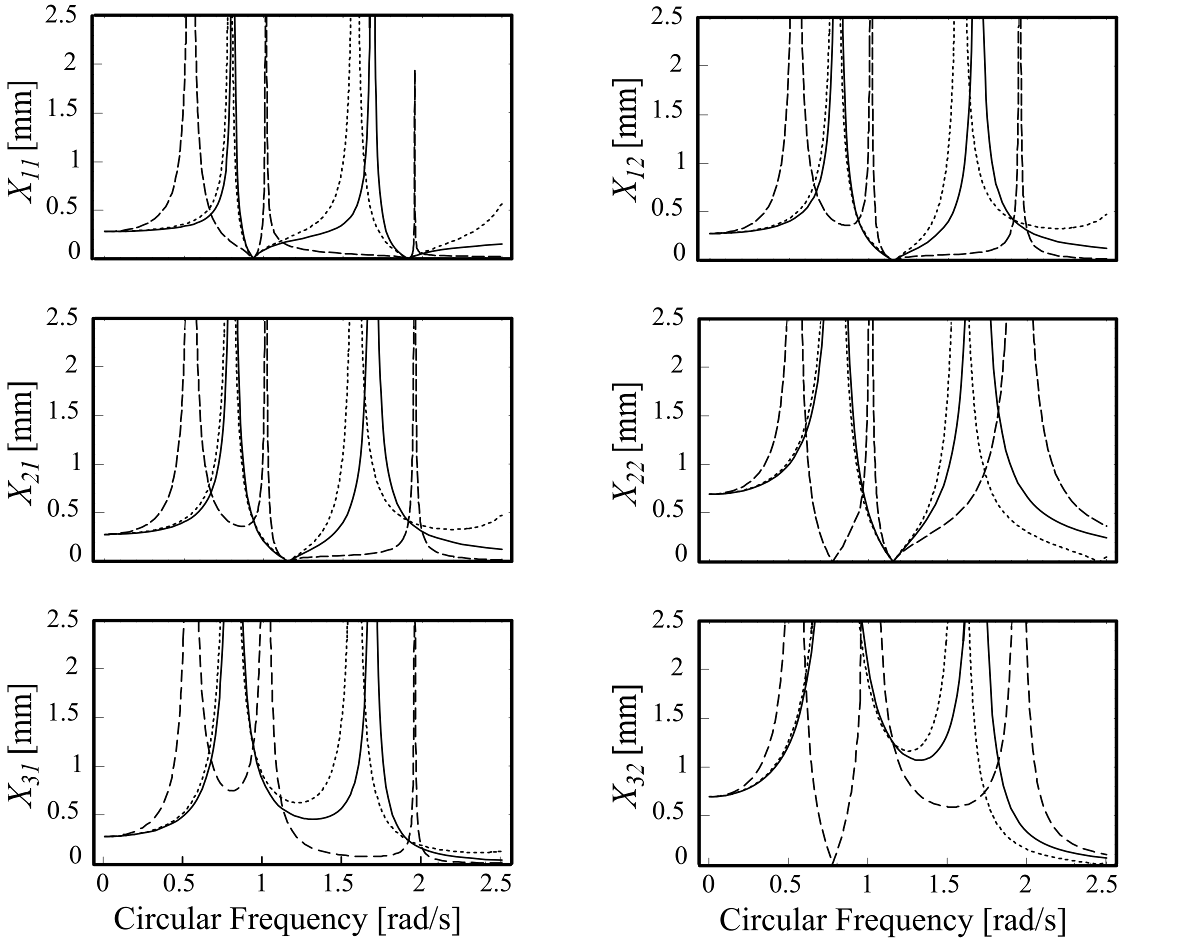

Fig. 3. Frequency responses X_{ij} for the system shown in Fig. 1 for the arbitrary selected parameters: $k_1 = 36$ N/m, $k_2 = 24$ N/m, $k_3 = 8$ N/m, $m_2 = 10$ kg, $m_3 = 6$ kg, $P_0 = 0.01$ N and different values of m_1. (——) $m_1 = 2$ kg, (······) $m_1 = 10$ kg, (-----) $m_1 = 100$ kg. $\omega_{10} = 0.935$ rad/s, $\omega_{11} = 1.913$ rad/s.

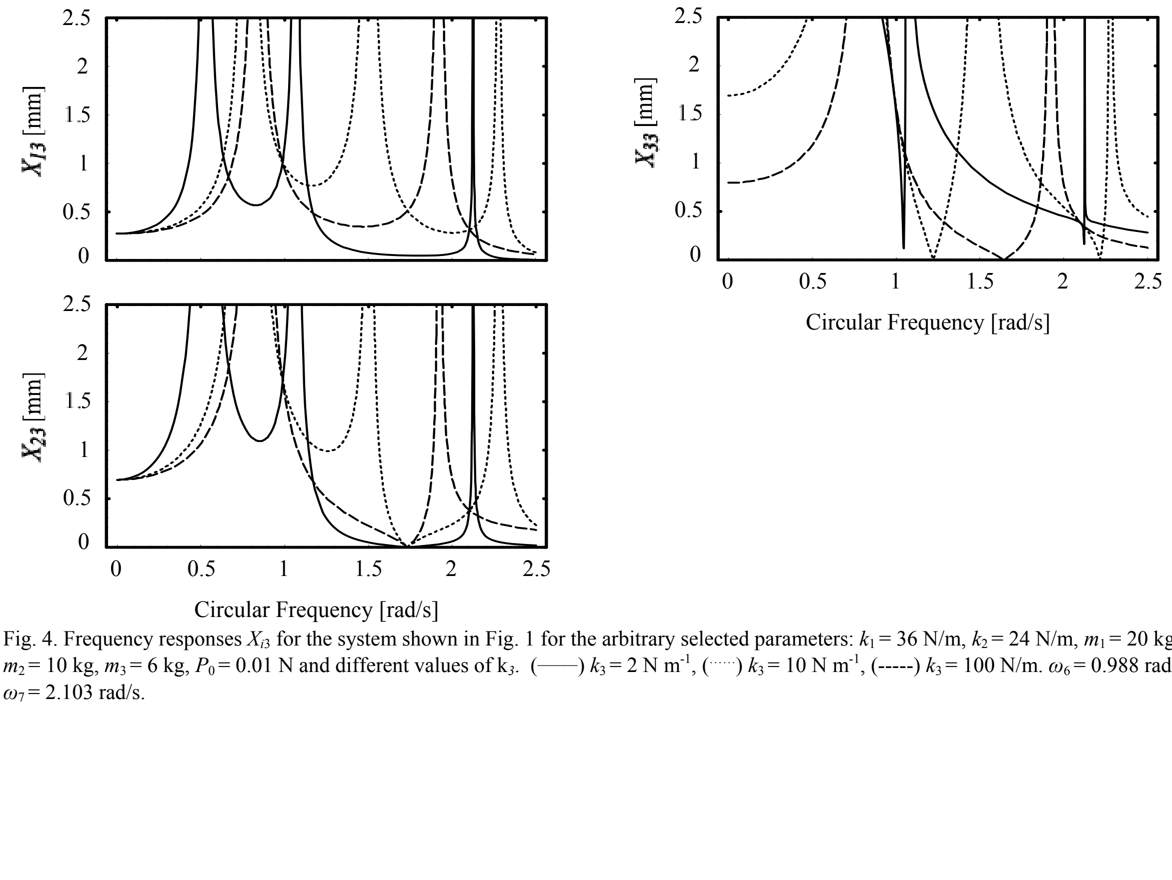

Fig. 4. Frequency responses X_{i3} for the system shown in Fig. 1 for the arbitrary selected parameters: $k_1 = 36$ N/m, $k_2 = 24$ N/m, $m_1 = 20$ kg, $m_2 = 10$ kg, $m_3 = 6$ kg, $P_0 = 0.01$ N and different values of k_3. (——) $k_3 = 2$ N m^{-1}, (······) $k_3 = 10$ N m^{-1}, (-----) $k_3 = 100$ N/m. $\omega_6 = 0.988$ rad/s, $\omega_7 = 2.103$ rad/s.

$$A_{32} = \frac{m_1^2 k_3 P_0}{k_1[k_1 m_2 m_3 - k_3 m_1(m_2 + m_3)]} \quad (23)$$

By the force application on m_2, the amplitude of m_2 becomes $A_{22} = A_{33}$ because of the symmetry of the system matrices. The amplitude of mass m_3 is then

$$A_{33} = \frac{m_1(k_3 m_1 - k_1 m_2)P_0}{k_1[k_1 m_2 m_3 - k_3 m_1(m_2 + m_3)]} \quad (24)$$

By the force application on mass m_1 at the frequency ω_2, the fixed point of X_{11} vanishes whereas X_{21} and X_{31} stay keeping their fixed points at ω_2 because of the symmetry of the mass and stiffness matrices of the whole system. The amplitudes of masses m_2 and m_3 are then $A_{21} = A_{12}$ and $A_{31} = A_{13}$, respectively.

By force application on mass m_1 at the frequency ω_5, the natural frequency of the subsystem *S7*, the subsystem *S7* vibrate in its second mode shape with an amplitude ratio $A_{31}/A_{21} = m_2/m_3$. The system *S1* vibrates at this frequency with the amplitude

$$A_{11} = \frac{P_0}{k_1 - m_1\omega_5^2} \quad (25)$$

In order to remain the spring k_2 undeformed, the mass m_2 vibrates with the same amplitude and in the same direction as m_1 as shown in Table 1, 8th column, that is

$$A_{21} = \frac{P_0}{k_1 - m_1\omega_5^2} \quad (26)$$

For the amplitude of mass m_3 we get

$$A_{31} = \frac{m_2}{m_3} A_{21} = \frac{m_2}{m_3} \frac{P_0}{k_1 - m_1\omega_5^2} \quad (27)$$

Also because of the symmetry of the system matrices **M** and **K**, the amplitudes of mass m_1 by the application of the force on m_2 or m_3 become $A_{13} = A_{31}$, and $A_{12} = A_{21}$, respectively, and its frequency responses X_{13} and X_{12} have stiffness fixed points at ω_5.

3.5. Fixed points at the frequencies ω3 and ω4

3.5.1. Force application on m2

If the force P acts on mass m_2 in Fig.1 with an excitation frequency equal to one of the eigenfrequencies ω_4 and ω_3, of subsystems *S2* and *S3*, respectively, shown in Fig. 2, then these subsystems act as internal absorbers for mass m_2 as shown in Fig. 5.

Absorber S2 at the frequency ω4

This case occurs when the mass m_2 is acted upon a force with the natural frequency ω_4 of the subsystem *S2*. At this frequency, the subsystem (m_2, k_3, m_3) remains at rest ($x_2 = x_3 = 0$), where the internal absorber *S2* vibrates at its natural frequency with the constant amplitude $A_{12} = P_0/k_2$, which is obtained from the force balance at m_2; that is, from $k_2A_{12}\cos\omega_4 t = P_0\cos\omega_4 t$, follows: $A_{12} = P_0/k_2$. Also the frequency responses X_{i2}, $_{i=1,2,3}$ have mass and stiffness fixed points at the frequency ω_4 since the amplitudes A_{i2} are independent of m_2, m_3, and k_3 at this frequency as shown in Table 1.

Absorber S3 at the frequency ω_3

If the excitation frequency of the applied load becomes equal to ω_3, the natural frequency of the subsystem S3, then this subsystem vibrates with a constant amplitude A_{32} at its natural frequency, where the subsystem (k_1, k_2, m_1, m_2) remains at rest ($x_1 = x_2 = 0$). Therefore at the frequency ω_3 all three frequency responses X_{i2} have mass and stiffness fixed points by varying the values of k_1, k_2, m_1, or m_2. The vibration amplitude A_{32} of mass m_3 follows from the force balance at m_2. (from $k_3A_{32}\cos\omega_3 t = P_0\cos\omega_3 t$ follows : $A_{32} = P_0/k_3$).

Also the frequency responses X_{2i}, $_{i=1,2,3}$ have always fixed points at the absorber frequencies ω_3 and ω_4 independent of which mass, the force acts, because of the symmetry of the system matrices. However, the amplitude of m_2 may become nonzero when the force is applied on the other masses.

3.5.2. Force applied on mass m1

When the force P acts on mass m_1 at the frequency ω_3, the natural frequency of the top subsystem *S3* (k_3-m_3-system), then the forces acting on mass m_2 will be balanced ($k_3x_3 = k_2x_1$), since *S3* serves at this frequency as an absorber for mass m_2. Therefore, at the frequency ω_3 mass m_2 remains at rest independent of the values of k_1, k_2, m_1, and m_2, whereas the masses m_1 and m_3 vibrate out of phase with different but proportional amplitudes.

Using the theory of vibration of single degree of freedom systems [8], we obtain for the amplitude of mass m_1

$$A_{11} = \frac{P_0}{k_1 + k_2 - m_1\omega_3^2} = \frac{m_3 P_0}{k_3 m_1 - (k_1 + k_2)m_3} \quad (28)$$

From the force balance at mass m_2 where $k_3x_3 = k_2x_1$, we obtain for the vibration amplitude of mass m_3

$$A_{31} = \frac{k_2}{k_3} A_{11} = \frac{k_2}{k_3} \frac{m_3 P_0}{[k_3 m_1 - (k_1 + k_2)m_3]} \quad (29)$$

From Eqs. (28) and (29) it is observable that the amplitudes of the masses m_1 and m_3 are independent of m_2 at the frequency ω_3. Therefore their frequency responses X_{11} and X_{31} possess mass fixed point at ω_3 by varying the values of m_2.

Also if the force acts on mass m_1 at the frequency ω_4, then the amplitude of m_2 becomes $A_{21} = A_{12} = P_0/k_2$ because of symmetry of the mass and stiffness matrices of the entire system. The amplitude of m_3 becomes in this case

$$A_{32} = \frac{k_3}{k_2} \frac{P_0}{k_3 - m_3\omega_4^2} = \frac{k_3}{k_2} \frac{m_1 P_0}{[k_3 m_1 - (k_1 + k_2)m_3]} \quad (30)$$

which is independent of mass m_2. Hence the frequency response X_{31} has a mass fixed point at ω_4 by varying the values of m_2. However, the mass fixed point of the frequency response X_{11} vanishes in this case.

3.5.3. Force applied on mass m3

When the force P acts on the mass m_3 at the frequency ω_4, the natural frequency of the bottom subsystem *S2* (k_1-

k_2-m_1-system), then the forces acting on mass m_2 will be balanced ($k_3x_3 = k_2x_1$), since *S2* serves in this case as an

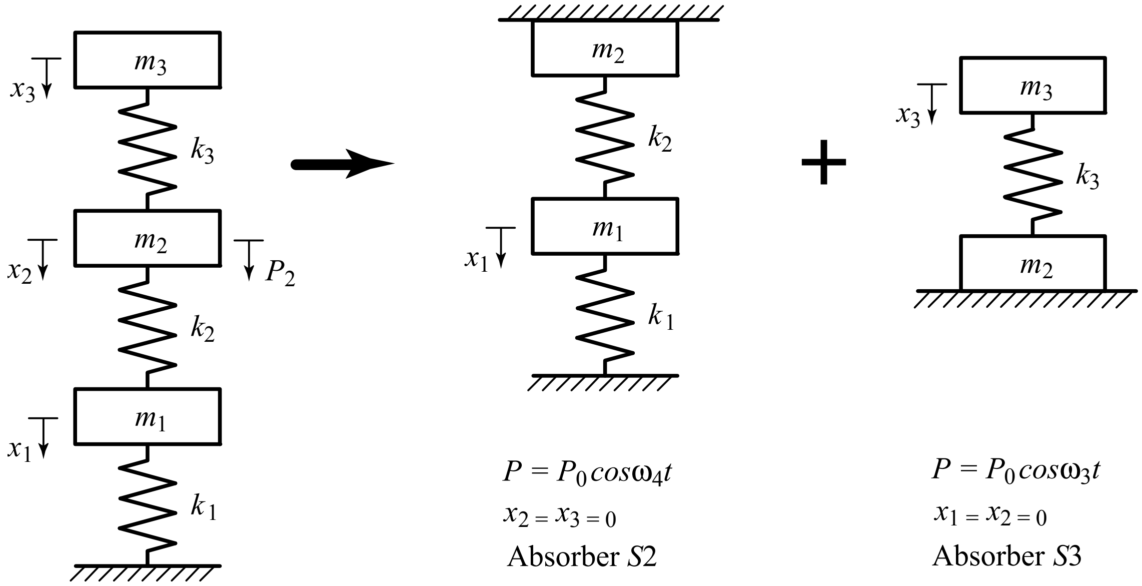

Fig. 5. Entire and subsystems (as absorbers).

internal absorber for mass m_2. Therefore at this frequency, m_2 remains at rest independent of the values of k_3, m_2, and m_3, whereas the masses m_1 and m_3 vibrate out of phase with different but proportional amplitudes. The vibration amplitude of mass m_3 may obtained from the subsystem *S3* as

$$A_{33} = \frac{P_0}{k_3 - m_3\omega_4^2} = \frac{m_1 P_0}{k_3 m_1 - (k_1 + k_2)m_3} \quad (31)$$

From the force balance on mass m_2 where $k_3x_3 = k_2x_1$ we get for the vibration amplitude of mass m_1:

$$A_{13} = \frac{k_3}{k_2} A_{33} = \frac{k_3}{k_2} \frac{m_1 P_0}{[k_3 m_1 - (k_1 + k_2)m_3]} \quad (32)$$

From Eqs. (31) and (32) it is obvious that the amplitudes of the masses m_1 and m_3 are independent of mass m_2 at the frequency ω_4. Therefore their frequency responses X_{13} and X_{33} have mass fixed points at ω_4 by varying the values of m_2. Also at the frequency ω_3, the amplitude of mass m_1 becomes

$$A_{13} = \frac{k_2}{k_3} \frac{P0}{[k_1 + k_2 - m_1\omega_3^2]} = \frac{k_2}{k_3} \frac{m_3 P_0}{[k_3 m_1 - (k_1 + k_2)m_3]} \quad (33)$$

which is independent of mass m_2. Hence the frequency response X_{13} has a mass fixed point at ω_3 by varying the values of m_2. On the other hand, the mass fixed point of the frequency response X_{33} vanishes in this case.

3.6. Fixed points at the frequency ω_1 (static case)

The static deflections $A_{s,i}$ of the masses m_1, m_2, and m_3 due to the force amplitude P_0 are defined as follows:

The force acts on mass m_1

$$A_{s,1} = A_{s,2} = A_{s,3} = \frac{P_0}{k_1} \quad (34)$$

The force acts on mass m_2

$$A_{s,1} = \frac{P_0}{k_1} \quad (35)$$

$$A_{s,2} = A_{s,3} = \frac{(k_1 + k_2)P_0}{k_1 k_2} \quad (36)$$

The force acts on mass m_3

$$A_{s,1} = \frac{P_0}{k_1} \quad (37)$$

$$A_{s,2} = \frac{(k_1 + k_2)P_0}{k_1 k_2} \quad (38)$$

$$A_{s,3} = \frac{(k_1 k_2 + k_1 k_3 + k_2 k_3)P_0}{k_1 k_2 k_3} \quad (39)$$

From Eqs. (34) through (39) it is obvious that the static deflections A_{si} are independent of the masses m_i, i=1,2,3. Therefore all frequency responses X_{ij}, i,j=1,2,3 have mass fixed points at the frequency $\omega_1 = 0$. Since all static deflections are dependent on the stiffness k_1, varying the values of this stiffness will not lead to stiffness fixed points in all frequency responses X_{ij}.

By force application on mass m_1, all frequency responses X_{i1} have stiffness fixed points by varying the values of k_2 or k_3.

By force application on mass m_2, all frequency responses X_{i2} have stiffness fixed points at ω_1 when the values of k_3 are varied. Also only X_{12} possesses a stiffness fixed point by varying the values of k_2.

Acts the force on the mass m_3, then the frequency response X_{13} has stiffness fixed points at ω_1 by varying the values of k_2 or k_3 and X_{23} has a stiffness fixed point by varying k_3.The response X_{33} has no fixed points at ω_1 in this case.

4. Conclusions

In this paper, the physical nature of mass and stiffness fixed points of undamped linear discrete vibrational systems is explored. It is found that the mass and stiffness fixed points frequencies of these systems represent natural frequencies of subsystems of the entire system. Furthermore, it is found that these fixed points are strongly related with the phenomenon of internal absorber and can be used to design vibratory systems with zero or constant amplitudes for some masses of the system. Also the mass and stiffness fixed points frequencies and their amplitudes of a system with three degree of freedom were determined and discussed in detail.

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Appendix

$$C_1 = \frac{P_0}{k_1} \tag{40}$$

$$C_2 = \frac{(k_1 + k_2)P_0}{k_1 k_2} \tag{41}$$

$$C_3 = \frac{(k_1 k_2 + k_1 k_3 + k_2 k_3)P_0}{k_1 k_2 k_3} \tag{42}$$

$$C_4 = \frac{P_0}{k_2} \tag{43}$$

$$C_5 = \frac{P_0}{k_3} \tag{44}$$

$$C_6 = \frac{m_1 P_0}{k_3 m_1 - (k_1 + k_2)m_3} \tag{45}$$

$$C_7 = \frac{k_3}{k_2} \frac{m_1 P_0}{[k_3 m_1 - (k_1 + k_2)m_3]} \tag{46}$$

$$C_8 = \frac{m_3 P_0}{k_3 m_1 - (k_1 + k_2)m_3} \tag{47}$$

$$C_9 = \frac{k_2}{k_3} \frac{m_3 P_0}{[k_3 m_1 - (k_1 + k_2)m_3]} \tag{48}$$

$$C_{10} = \frac{m_2 m_3 P_0}{k_1 m_2 m_3 - k_3 m_1 (m_2 + m_3)} \tag{49}$$

$$C_{11} = \frac{m_2^2 P_0}{k_1 m_2 m_3 - k_3 m_1 (m_2 + m_3)} \tag{50}$$

$$C_{12} = \frac{m_1 (m_1 k_3 - k_1 m_3) P_0}{k_1 [k_1 m_2 m_3 - k_3 m_1 (m_2 + m_3)]} \tag{51}$$

$$C_{13} = \frac{m_1 (m_1 k_3 - k_1 m_2) P_0}{k_1 [k_1 m_2 m_3 - k_3 m_1 (m_2 + m_3)]} \tag{52}$$

$$C_{14} = \frac{k_3 m_1^2) P_0}{k_1 [k_1 m_2 m_3 - k_3 m_1 (m_2 + m_3)]} \tag{53}$$

$$C_{15} = \frac{k_3}{k_2} \frac{P_0}{[k_3 - m_3 \omega_{10}^2]} \tag{54}$$

$$C_{16} = \frac{k_3}{k_2} \frac{P_0}{[k_3 - m_3 \omega_{11}^2]} \tag{55}$$

$$C_{17} = \frac{k_2}{k_3} \frac{P_0}{[k_1 + k_2 - m_1 \omega_8^2]} \tag{56}$$

$$C_{18} = \frac{k_2}{k_3} \frac{P_0}{[k_1 + k_2 - m_1 \omega_9^2]} \tag{57}$$

$$C_{19} = \frac{2m_2^2}{m_3} \frac{P_0}{[m_2 (k_1 - k_2) - k_2 m_1 + C]} \tag{58}$$

$$C_{20} = \frac{2m_2^2}{m_3} \frac{P_0}{[m_2 (k_2 - k_1) + k_2 m_1 + C]} \tag{59}$$

$$C_{21} = \frac{m_2}{k_2 m_3} \frac{[m_2 (k_2 + k_1) - k_2 m_1 + C]}{[m_2 (k_2 - k_1) + k_2 m_1 - C]} P_0 \tag{60}$$

$$C_{22} = \frac{m_2}{k_2 m_3} \frac{[m_2 (k_2 + k_1) - k_2 m_1 - C]}{[m_2 (k_2 - k_1) + k_2 m_1 + C]} P_0 \tag{61}$$

$$C = \sqrt{[k_1 m_2 + k_2 (m_1 + m_2)]^2 - 4 k_1 k_2 m_1 m_2} \tag{62}$$