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Finite Element Formulation of Internally Balanced Blatz – Ko Material Model

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Abstract

Material constitutive models often include internal variables in order to capture realistic mechanical effects such as viscosity. Recent work for compressible hyperelastic material is developed based on applying the argument of calculus variation to two-factor multiplicative decomposition of the deformation gradient. The finite element formulation for this new treatment is developed, however, the implementation sheds light on a special form of constitutive model. In particular, the material model is a function of the first and third invariants of new quantities derived from the counterparts of the multiplicative decomposition. These new quantities are defined in analogy to the right Cauchy Green tensor. This work demonstrates the required treatment for a special material model that is formulated using the second and third principal invariants of these new derived quantities. Mainly, the treatment simplifies the internal balance equation that emerges from the variational treatment. This facilitates the linearization procedure of this new formulation for internally balanced compressible hyperelastic material. The present work permits the future use of more complicated internally balanced hyperelastic models.

Keywords: Hyperelasticity; internal balance; finite element;

1. Introduction

Large deformation constitutive models can be expressed in terms of deformation gradient $\boldsymbol{F}$ multiplicative decompositions [1, 2, 3] such as

$$\boldsymbol{F} = \widehat{\boldsymbol{F}}\breve{\boldsymbol{F}}. \tag{1}$$

The usual treatment is that $\widehat{\boldsymbol{F}}$ models elastic response and it is associated to the rules of variational calculus. The $\breve{\boldsymbol{F}}$ portion then models inelastic response usually by means of a time dependent evolution law. This multiplicative decomposition serves to pertain particular portions of $\boldsymbol{F}$ to specific parts of the material response. It has been widely used in plasticity [4, 5, 6, 7], viscoelasticity [8, 9], growth and remodeling of biological tissues [10, 11, 12, 13]. The implementation procedure of these material models in the frame of nonlinear finite element is well established [14, 15, 16, 17].

A new scheme of viewing incompressible hyperelastic material response is introduced in [18, 19]. In fact, the arguments of variation are applied to both portions of the deformation gradient decomposition. The decomposition itself is determined on the basis of an additional internal balance equation that emerges naturally from the variational treatment. Further theoretical treatment for compressible hyperelastic model is presented in [20]. The total Lagrange formulation of this new treatment is derived by linearizing the achieved weak form with respect to both portions of multiplicative decomposition [21]. Modeling material response using similar procedure presented in this paper can be found in [22, 23, 24, 25, 26].

The demonstrated implementation in [21] for compressible hyperelastic material focused on a special form of internally balance Blatz – Ko material model $W(\hat{I}_1,\hat{I}_3,\breve{I}_1,\breve{I}_3)$. Here $\hat{I}_1$, $\hat{I}_2$ and $\hat{I}_3$ are the principal invariants of $\widehat{\boldsymbol{C}} = \widehat{\boldsymbol{F}}^T\widehat{\boldsymbol{F}}$ (and $\widehat{\boldsymbol{B}} = \widehat{\boldsymbol{F}}\widehat{\boldsymbol{F}}^T$) while $\breve{I}_1$, $\breve{I}_2$ and $\breve{I}_3$ are the principal invariants of $\breve{\boldsymbol{C}} = \breve{\boldsymbol{F}}^T\breve{\boldsymbol{F}}$ (and $\breve{\boldsymbol{B}} = \breve{\boldsymbol{F}}\breve{\boldsymbol{F}}^T$). These new quantities $\widehat{\boldsymbol{C}}$, $\breve{\boldsymbol{C}}$, $\widehat{\boldsymbol{B}}$ and $\breve{\boldsymbol{B}}$ are second order symmetric tensors and they are defined in analogy with $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ and $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$. This work sheds light on the mathematical treatment that is required to implement a material model that has the form of $W(\hat{I}_2,\hat{I}_3,\breve{I}_2,\breve{I}_3)$ in total Lagrange and update Lagrange formulations.

2. Continuum Mechanics

Strain energy function for isotropic hyperelastic material can be expressed in terms of the principal invariants such as $W(I_1,I_2,I_3)$.To implement this type of material model in total Lagrange formulation, it is required to derive two important quantities that are Second Piola – Kirchhoff stress tensor and material elasticity tensor. The second Piola – Kirchhoff $\boldsymbol{S}$ stress can be written as

$$\boldsymbol{S} = 2\left(\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{C}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \boldsymbol{C}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \boldsymbol{C}}\right). \tag{2}$$

The fourth order material elasticity tensor $\mathbb{C}$ is obtained by

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$$\mathbb{C} = 2\,\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}} = 4\,\frac{\partial^2 W}{\partial \boldsymbol{C}\partial \boldsymbol{C}}, \tag{3}$$

a detailed definition for $\mathbb{C}$ in terms of principal invariants can be found in [27].

For Update Lagrange formulation, it is required to implement Cauchy stress and spatial elasticity tensors. Cauchy stress tensor $\boldsymbol{\sigma}$ is obtained using a Piola transformation of the second Piola Kirchhoff stress tensor such as

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T, \tag{4}$$

where $J = \det \boldsymbol{F}$. The Piola transformation [28] is a push forward operation that is scaled by J^{-1}. The spatial elasticity tensor $\boldsymbol{D}$ is defined in via Piola transformation of $\mathbb{C}$ [29, 27]. It is written in component form as

$$D_{abcd} = J^{-1}F_{aI}F_{bJ}F_{cK}F_{dL}\mathbb{C}_{IJKL}. \tag{5}$$

3. Internal Balance

A new scheme of viewing incompressible hyperelastic material response is introduced in [18]. The arguments of variation are applied to both portions of the deformation gradient decomposition. The decomposition itself is determined on the basis of an additional internal balance equation that emerges naturally from the variational treatment. A review of the compressible hyperelastic scheme [20] is summarized here. It has been shown that the second Piola – Kirchhoff $\boldsymbol{S}$ stress is obtained as consequence of applying the argument of variation with respect to $\boldsymbol{C}$. It can be written as

$$\boldsymbol{S}(\boldsymbol{C},\breve{\boldsymbol{C}}) = 2\left(\frac{\partial W}{\partial \hat{I}_1}\breve{\boldsymbol{C}}^{-1} + \frac{\partial W}{\partial \hat{I}_2}\left(\hat{I}_1\breve{\boldsymbol{C}}^{-1} - \boldsymbol{M}\right) + \frac{\partial W}{\partial \hat{I}_3}\hat{I}_3\boldsymbol{C}^{-1}\right), \tag{6}$$

where $\boldsymbol{M} = \breve{\boldsymbol{C}}^{-1}\boldsymbol{C}\breve{\boldsymbol{C}}^{-1}$. Notice that $\boldsymbol{S}$ is a function of $\boldsymbol{C}$ and $\breve{\boldsymbol{C}}$. The decomposition of the deformation gradient is found by solving an internal balance equation that arises from the variation with respect to $\breve{\boldsymbol{C}}$. The internal balance equation is

$$\boldsymbol{\Psi} = \boldsymbol{0}, \tag{7}$$

where $\boldsymbol{\Psi}$ is an internal balance tensor

$$\boldsymbol{\Psi} = \hat{\boldsymbol{\Psi}}_1 + \hat{\boldsymbol{\Psi}}_2 + \hat{\boldsymbol{\Psi}}_3 + \breve{\boldsymbol{\Psi}}_1 + \breve{\boldsymbol{\Psi}}_2 + \breve{\boldsymbol{\Psi}}_3 \tag{8}$$

with individual parts

$$\hat{\boldsymbol{\Psi}}_1 = 2\frac{\partial W}{\partial \hat{I}_1}\frac{\partial \hat{I}_1}{\partial \breve{\boldsymbol{C}}} = -2\frac{\partial W}{\partial \hat{I}_1}\boldsymbol{M}, \tag{9}$$

$$\hat{\boldsymbol{\Psi}}_2 = 2\frac{\partial W}{\partial \hat{I}_2}\frac{\partial \hat{I}_2}{\partial \breve{\boldsymbol{C}}} = 2\frac{\partial W}{\partial \hat{I}_2}\left(\boldsymbol{N} - \hat{I}_1\boldsymbol{M}\right), \tag{10}$$

$$\hat{\boldsymbol{\Psi}}_3 = 2\frac{\partial W}{\partial \hat{I}_3}\frac{\partial \hat{I}_3}{\partial \breve{\boldsymbol{C}}} = -2\frac{\partial W}{\partial \hat{I}_3}\hat{I}_3\breve{\boldsymbol{C}}^{-1}, \tag{11}$$

$$\breve{\boldsymbol{\Psi}}_1 = 2\frac{\partial W}{\partial \breve{I}_1}\frac{\partial \breve{I}_1}{\partial \breve{\boldsymbol{C}}} = 2\frac{\partial W}{\partial \breve{I}_1}\boldsymbol{I}, \tag{12}$$

$$\breve{\boldsymbol{\Psi}}_2 = 2\frac{\partial W}{\partial \breve{I}_2}\frac{\partial \breve{I}_2}{\partial \breve{\boldsymbol{C}}} = 2\frac{\partial W}{\partial \breve{I}_2}\left(\breve{I}_1\boldsymbol{I} - \breve{\boldsymbol{C}}\right), \tag{13}$$

$$\breve{\boldsymbol{\Psi}}_3 = 2\frac{\partial W}{\partial \breve{I}_3}\frac{\partial \breve{I}_3}{\partial \breve{\boldsymbol{C}}} = 2\frac{\partial W}{\partial \breve{I}_3}\breve{I}_3\breve{\boldsymbol{C}}^{-1}, \tag{14}$$

where $\boldsymbol{N} = \breve{\boldsymbol{C}}^{-1}\boldsymbol{C}\breve{\boldsymbol{C}}^{-1}\boldsymbol{C}\breve{\boldsymbol{C}}^{-1}$ and $\boldsymbol{I}$ is the second order identity tensor. Notice that $\hat{\boldsymbol{\Psi}}_2$ is relatively complicated compared to other individual parts of $\boldsymbol{\Psi}$. It is tedious task to differentiate $\hat{\boldsymbol{\Psi}}_2$ with respect to $\boldsymbol{C}$ and $\breve{\boldsymbol{C}}$ during linearization procedure. This leads to avoid the use of strain energy function based on $\hat{I}_2$. To overcome this difficulties, the following procedure is applied to simplify the expression of $\hat{\boldsymbol{\Psi}}_2$. First of all, the push forward operation is performed

$$\boldsymbol{F}\frac{\partial \hat{I}_2}{\partial \breve{\boldsymbol{C}}}\boldsymbol{F}^T = \boldsymbol{F}\left(\boldsymbol{N} - \hat{I}_1\boldsymbol{M}\right)\boldsymbol{F}^T = \hat{\boldsymbol{B}}^3 - \hat{I}_1\hat{\boldsymbol{B}}^2, \tag{15}$$

then Cayley – Hamilton equation is applied to simplify (15) to

$$\boldsymbol{F}\frac{\partial \hat{I}_2}{\partial \breve{\boldsymbol{C}}}\boldsymbol{F}^T = \hat{I}_3\boldsymbol{I} - \hat{I}_2\hat{\boldsymbol{B}}, \tag{16}$$

and finally (16) is pulled backward to get

$$\frac{\partial \hat{I}_2}{\partial \breve{\boldsymbol{C}}} = \boldsymbol{F}^{-1}\left(\hat{I}_3\boldsymbol{I} - \hat{I}_2\hat{\boldsymbol{B}}\right)\boldsymbol{F}^{-T} = \hat{I}_3\boldsymbol{C}^{-1} - \hat{I}_2\breve{\boldsymbol{C}}^{-1}, \tag{17}$$

now $\hat{\boldsymbol{\Psi}}_2$ has more practical expression by virtue of (17) such as

$$\hat{\boldsymbol{\Psi}}_2 = 2\frac{\partial W}{\partial \hat{I}_2}\frac{\partial \hat{I}_2}{\partial \breve{\boldsymbol{C}}} = 2\frac{\partial W}{\partial \hat{I}_2}\left(\hat{I}_3\boldsymbol{C}^{-1} - \hat{I}_2\breve{\boldsymbol{C}}^{-1}\right). \tag{18}$$

For updated Lagrange, the Cauchy stress for internally balanced scheme is defined by push forward of (6) as

$$\boldsymbol{\sigma}(\boldsymbol{B},\hat{\boldsymbol{B}}) = \frac{2}{J}\left(\frac{\partial W}{\partial \hat{I}_1}\hat{\boldsymbol{B}} + \frac{\partial W}{\partial \hat{I}_2}\left(\hat{I}_1\hat{\boldsymbol{B}} - \hat{\boldsymbol{B}}^2\right) + \frac{\partial W}{\partial \hat{I}_3}\hat{I}_3\boldsymbol{I}\right). \tag{19}$$

Similarly, the internal balance equation (7) becomes

$$\boldsymbol{\Xi}(\boldsymbol{B},\hat{\boldsymbol{B}}) = \boldsymbol{F}\boldsymbol{\Psi}\boldsymbol{F}^T = \boldsymbol{0}, \tag{20}$$

where $\boldsymbol{\Xi}$ is the internal balance tensor with the individual parts

$$\boldsymbol{\Xi} = 2\left(\hat{\boldsymbol{\Xi}}_1 + \hat{\boldsymbol{\Xi}}_2 + \hat{\boldsymbol{\Xi}}_3 + \breve{\boldsymbol{\Xi}}_1 + \breve{\boldsymbol{\Xi}}_2 + \breve{\boldsymbol{\Xi}}_3\right), \tag{21}$$

these individual parts of $\boldsymbol{\Xi}$ are listed in the appendix.

4. Blatz – Ko Model

The generalized Blatz – Ko [30, 31, 32] can be written as

$$W(I_1, I_2, I_3) = \frac{\mu(1-f)}{2}\left(\frac{I_2}{I_3} + \frac{1}{\alpha}I_3^{\alpha} - \zeta\right) + \frac{\mu f}{2}\left(\frac{1}{\alpha}I_3^{-\alpha} + I_1 - \zeta\right), \tag{22}$$

where $\alpha = \nu/(1-2\nu)$ and $\zeta = 1 + 1/\nu$. Blatz – Ko model has three material parameters namely Poisson's ratio ν, shear modulus μ and volume fraction of voids in foam rubber material f. In this work a special case of (22) is achieved [30, 31, 33] by applying $\nu = 1/4$ and $f = 0$ such as

$$W_{BK}(I_2, I_3) = \frac{\mu}{2}\left(\frac{I_2}{I_3} + 2I_3^{1/2} - 5\right). \tag{23}$$

Substituting (23) into (2) to get Second Piola – Kirchhoff stress

$$\boldsymbol{S}_{BK} = \mu\left(\frac{1}{I_3}(I_1\boldsymbol{I} - \boldsymbol{C}) + \mathcal{L}\boldsymbol{C}^{-1}\right) = \mu(J\boldsymbol{C}^{-1} - \boldsymbol{C}^{-2}), \tag{24}$$

where $\mathcal{L} = -I_2/I_3 + I_3^{1/2}$. The elasticity tensor is obtained by substituting (23) into (3)

$$\mathbb{C}_{BK} = \mu J(\boldsymbol{C}^{-1}\otimes\boldsymbol{C}^{-1} - 2\boldsymbol{C}^{-1}\odot\boldsymbol{C}^{-1}) + 2\mu(\boldsymbol{C}^{-2}\odot\boldsymbol{C}^{-1} + \boldsymbol{C}^{-1}\odot\boldsymbol{C}^{-2}), \tag{25}$$

where $\otimes$ is the dyadic operator and $\odot$ operator has the same definition given in [27], further details can be found in the appendix. Now substitute (24) into (4) to obtain Cauchy stress

$$\boldsymbol{\sigma}_{BK} = \mu(\boldsymbol{I} - J^{-1}\boldsymbol{B}^{-1}). \tag{26}$$

The spatial tensor of elasticity is obtained by the virtue of (5) as

$$\boldsymbol{D}_{BK} = \mu(\boldsymbol{I}\otimes\boldsymbol{I} - 2\boldsymbol{I}\odot\boldsymbol{I}) + 2\mu J^{-1}(\boldsymbol{B}^{-1}\odot\boldsymbol{I} + \boldsymbol{I}\odot\boldsymbol{B}^{-1}). \tag{27}$$

5. Blatz – Ko Internal Balance Model

The form of internally balanced material model is motivated by special case of Blatz – Ko model

$$W_{IB} = \frac{\mu(1+\beta)}{2}\left(\frac{\hat{I}_2}{\hat{I}_3} + 2\hat{I}_3^{1/2} - 5\right) + \frac{\mu(1+\beta)}{2\beta}\left(\frac{\breve{I}_2}{\breve{I}_3} + 2\breve{I}_3^{1/2} - 5\right), \quad (28)$$

where β is a positive material parameter that quantify the contribution of two – factor multiplicative decomposition (1). The limits $\beta \to 0$ and $\beta \to \infty$ retrieve the hyperelastic behavior in full nonlinear strain range [20]. An equivalent form of (28) is used in [34] to investigate uniaxial loading by solving nonlinear boundary value problem. In this work, the finite element formulation of (28) is presented that permits general loading scenarios.

The Piola – Kirchhoff stress of internally balanced material (28) is obtained by (6)

$$\boldsymbol{S}_{IB} = \bar{\mu}\left(\hat{\mathcal{L}}\boldsymbol{C}^{-1} + \frac{1}{\hat{I}_3}\left(\hat{I}_1\breve{\boldsymbol{C}}^{-1} - \boldsymbol{M}\right)\right), \quad (29)$$

where $\bar{\mu} = \mu(1+\beta)$ and $\hat{\mathcal{L}} = -\hat{I}_2/\hat{I}_3 + \hat{I}_3^{1/2}$. The stress tensor $\boldsymbol{S}_{IB}$ is coupled to internal balance equation $\boldsymbol{\Psi} = \boldsymbol{0}$ that is achieved by the virtue of equations (9),(11) – (14) and (18)

$$\boldsymbol{C}^{-1} - \hat{J}\breve{\boldsymbol{C}}^{-1} + \frac{1}{\beta}\left(\breve{J}\breve{\boldsymbol{C}}^{-1} - \breve{\boldsymbol{C}}^{-2}\right) = \boldsymbol{0}, \quad (30)$$

where $\hat{J} = \hat{I}_3^{1/2}$ and $\breve{J} = \breve{I}_3^{1/2}$. This quiet form of internal balance equation is obtained by using the simplified form of $\widehat{\boldsymbol{\Psi}}_2$ in (18) instead of (10) and further manipulation using Cayley – Hamilton equation. The push forward of (29) and the use of Cayley – Hamilton equation give a simplified form of Cauchy stress as

$$\boldsymbol{\sigma}_{IB} = \frac{\bar{\mu}}{J}\left(\hat{J}\boldsymbol{I} - \widehat{\boldsymbol{B}}^{-1}\right) \quad (31)$$

and it is coupled to $\boldsymbol{\Xi} = \boldsymbol{0}$ that is

$$\boldsymbol{I} - \hat{J}\widehat{\boldsymbol{B}} + \frac{1}{\beta}\left(\breve{J}\widehat{\boldsymbol{B}} - \widehat{\boldsymbol{B}}\boldsymbol{B}^{-1}\widehat{\boldsymbol{B}}\right) = \boldsymbol{0}. \quad (32)$$

It has been noticed that Piola – Kirchhoff stress (29) has still complicated form, a simplified form of (29) is required to avoid lengthy linearization procedure. To achieve that (32) is multiplied by $\widehat{\boldsymbol{B}}^{-1}$, then it is rearranged as

$$\hat{J}\boldsymbol{I} - \widehat{\boldsymbol{B}}^{-1} = \frac{1}{\beta}\left(\breve{J}\mathbf{I} - \widehat{\boldsymbol{B}}\boldsymbol{B}^{-1}\right), \quad (33)$$

then substitute (33) in (31) to obtain another and equivalent form of Cauchy stress such as

$$\boldsymbol{\sigma}_{IB} = \frac{\bar{\mu}}{J}\left(\hat{J}\boldsymbol{I} - \widehat{\boldsymbol{B}}^{-1}\right) = \frac{\bar{\mu}}{\beta J}\left(\breve{J}\mathbf{I} - \widehat{\boldsymbol{B}}\boldsymbol{B}^{-1}\right), \quad (34)$$

finally pull back the new form of Cauchy stress to get

$$\boldsymbol{S}_{IB} = \frac{\mu(1+\beta)}{\beta}\left(\breve{J}\boldsymbol{C}^{-1} - \boldsymbol{C}^{-1}\breve{\boldsymbol{C}}^{-1}\right). \quad (35)$$

Notice that Piola – Kirchhoff stress (35) has relatively simple form compared to (29).

6. Linearization

The finite element formulation of internally balanced compressible hyperelastic material is demonstrated in [21]. It is based on calculating a condensed fourth order elasticity tensor $\mathbb{C}_{con}$ that is defined as

$$\mathbb{C}_{con} = \mathbb{C}_{\boldsymbol{C}} - \mathbb{C}_{\breve{\boldsymbol{C}}} : \boldsymbol{\Psi}_{\breve{\boldsymbol{C}}}^{-1} : \boldsymbol{\Psi}_{\boldsymbol{C}}. \quad (36)$$

The tensors $\mathbb{C}_{\boldsymbol{C}}$ and $\mathbb{C}_{\breve{\boldsymbol{C}}}$ are obtained by differentiating second Piola – Kirchhoff stress (6) with respect to $\boldsymbol{C}$ and $\breve{\boldsymbol{C}}$, respectively. The terms $\boldsymbol{\Psi}_{\boldsymbol{C}}$ and $\boldsymbol{\Psi}_{\breve{\boldsymbol{C}}}$ are obtained by differentiating internal balance tensor (8) with respect to $\boldsymbol{C}$ and $\breve{\boldsymbol{C}}$, respectively.

Concerning the material model of interest in this work (28), these individual parts of (36) become

$$\mathbb{C}_{\boldsymbol{C}} = \frac{\bar{\mu}}{\beta}\left(\boldsymbol{G}\odot\boldsymbol{C}^{-1} + \boldsymbol{C}^{-1}\odot\boldsymbol{G} - 2\breve{J}\boldsymbol{C}^{-1}\odot\boldsymbol{C}^{-1}\right), \quad (37)$$

$$\mathbb{C}_{\breve{\boldsymbol{C}}} = \frac{\bar{\mu}}{\beta}\left(\breve{J}\boldsymbol{C}^{-1}\otimes\breve{\boldsymbol{C}}^{-1} + \breve{\boldsymbol{C}}^{-1}\odot\boldsymbol{G} + \boldsymbol{G}\odot\breve{\boldsymbol{C}}^{-1}\right), \quad (38)$$

$$\boldsymbol{\Psi}_{\boldsymbol{C}} = -2\boldsymbol{C}^{-1}\odot\boldsymbol{C}^{-1} - \hat{J}\breve{\boldsymbol{C}}^{-1}\otimes\boldsymbol{C}^{-1}, \quad (39)$$

$$\boldsymbol{\Psi}_{\breve{\boldsymbol{C}}} = \bar{J}_1\,\breve{\boldsymbol{C}}^{-1}\otimes\breve{\boldsymbol{C}}^{-1} + 2\bar{J}_2\,\breve{\boldsymbol{C}}^{-1}\odot\breve{\boldsymbol{C}}^{-1} + \frac{2}{\beta}\left(\breve{\boldsymbol{C}}^{-2}\odot\breve{\boldsymbol{C}}^{-1} + \breve{\boldsymbol{C}}^{-1}\odot\breve{\boldsymbol{C}}^{-2}\right), \quad (40)$$

where $\boldsymbol{G} = \breve{\boldsymbol{C}}^{-1}\boldsymbol{C}^{-1}$, $\bar{J}_1 = \hat{J} + \breve{J}/\beta$ and $\bar{J}_2 = \hat{J} - \breve{J}/\beta$. The spatial condensed elasticity tenor $\boldsymbol{D}_{con}$ for the updated Lagrange formulation can be obtained either by pushing forward of $\mathbb{C}_{con}$ or by performing push forward operation on each individual parts (37) – (40) then apply condensation procedure.

7. Solving Internal Balance Equation

It is an essential step to solve the internal balance equation in order to calculate the stress and elasticity tensor. This is carried out by solving (30) for $\breve{\boldsymbol{C}}$ given the value of $\boldsymbol{C}$ or solving (32) for $\widehat{\boldsymbol{B}}$ given the value of $\boldsymbol{B}$. Here it is chosen to solve (32). The multiplication of (32) by $\boldsymbol{B}$, then rearranging it, gives a simplified form of the internal balance equation

$$\widehat{\boldsymbol{B}}^2 + \left(\beta\hat{J} - \frac{J}{\hat{J}}\right)\widehat{\boldsymbol{B}}\boldsymbol{B} - \beta\boldsymbol{B} = \boldsymbol{0}. \quad (41)$$

It can be shown by further manipulation of (41) that the tensor $\boldsymbol{B}$ and $\widehat{\boldsymbol{B}}$ have the same orthogonal eigenvectors. Then, internal balance equation (41) can be expressed in principal frame using eigenvalues of $\boldsymbol{B}$ as $\lambda_1^2, \lambda_2^2, \lambda_3^2$ and eigenvalues of $\widehat{\boldsymbol{B}}$ as $\hat{\lambda}_1^2, \hat{\lambda}_2^2, \hat{\lambda}_3^2$ [35]. The internal balance equation in principal frame are expressed as a set nonlinear equations

$$\hat{\lambda}_1^4 + \beta\hat{\lambda}_1^3\hat{\lambda}_2\hat{\lambda}_3\lambda_1^2 - \frac{\lambda_1^2 J\hat{\lambda}_1}{\hat{\lambda}_2\hat{\lambda}_3} - \beta\lambda_1^2 = 0, \quad (42)$$

$$\hat{\lambda}_2^4 + \beta\hat{\lambda}_1\hat{\lambda}_2^3\hat{\lambda}_3\lambda_2^2 - \frac{\lambda_2^2 J\hat{\lambda}_2}{\hat{\lambda}_1\hat{\lambda}_3} - \beta\lambda_2^2 = 0, \quad (43)$$

$$\hat{\lambda}_3^4 + \beta\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3^3\lambda_3^2 - \frac{\lambda_3^2 J\hat{\lambda}_3}{\hat{\lambda}_1\hat{\lambda}_2} - \beta\lambda_3^2 = 0, \quad (44)$$

Newton – Raphson iterative procedure is used to solve (42) – (44).

8. Results and Discussion

The conventional hyperelastic Baltz – Ko model (23) and internally balanced Blatz – Ko model (28) are implemented in $FEAPpv^{©}$. It is in – house finite element package for updated Lagrange formulation written by Prof. R.L. Taylor, University of California at Berkeley. A cube of a unit length is meshed by 8 nodes brick element with three degrees of freedom per node. The cube is discretized by eight elements in total $(2 \times 2 \times 2)$. The homogeneous deformations namely uniaxial and simple shear are performed to investigate the response of the material models.

The uniaxial loading is given as $\boldsymbol{F} = \lambda_1\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \lambda_2\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \lambda_3\boldsymbol{e}_3 \otimes \boldsymbol{e}_3$ with corresponding stress $\boldsymbol{\sigma} = \sigma_{11}\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$ where $\lambda_1, \lambda_2, \lambda_3$ are principal stretches. The material is stretched in principal direction one and it is free to contract in the other principal directions. The achieved uniaxial stress σ_{11} for different values of β is shown in figure 1. It is verified that the achieved Cauchy stress value for given β coincides with achieved Cauchy stress value for $1/\beta$. Therefore, the values of $0 < \beta \leq 1$ are used in following discussion.

The achieved Cauchy stress curves for different values of β have a specific pattern up to particular value of stretch λ_1 that is stiffer than hyperelastic achieved Cauchy stress ($\beta = 0$). However, this pattern is not maintained for large value of stretch λ_1 see figure 2 (top). A better pattern is observed when Cauchy stress is normalized by $\mu(1+\beta)$ as shown in figure 2 (bottom). Now, the curves are ordered such as the stiffer response is achieved when $\beta = 0$ (hyperelastic) and the softest response is achieved for $\beta = 1$. The normalized Cauchy stress $\bar{\sigma}_{11} = \sigma_{11}/\mu(1+\beta)$ decreases as β increases from 0 to 1. The achieved Cauchy stress for $\beta = 0$ reaches an asymptotic value of one for large stretch $\lambda_1 \to \infty$; this is the retrieved value by (26). For internally balanced achieved results $\beta > 0$, the normalized Cauchy stress $\bar{\sigma}_{11}$ increases with stretching λ_1 up to a critical maximum value then it decreases. For the special case $\beta = 1$, this maximum value of normalized stress is found to be $\bar{\sigma}_{11} = 0.5824$ at $\lambda_1 = 6^{0.8} = 4.193$ this is shown in figure 2 (bottom).

The achieved finite element results are verified analytically by examining two special cases. The first case is for $\beta = 0$ this results in $\widehat{\boldsymbol{B}} \to \boldsymbol{B}$. This means that the hyperelastic Cauchy stress expressed in (26) is retrieved by internal balance Cauchy stress (31) at $\beta = 0$, the material parameter $\mu(\beta+1)$ becomes simply μ. The hyperelastic Cauchy stress is also retrieved at $\beta \to \infty$. Now, the normalized uniaxial stress has analytical expression such as $\bar{\sigma}_{11} = 1 - \lambda_1^{-5/2}$. The second special case is for $\beta = 1$. It can be shown that $\widehat{\boldsymbol{B}} = \boldsymbol{B}^{1/2} \to \hat{J} = J^{1/2}$ is a solution for (41). Then, corresponding normalized uniaxial stress becomes $\bar{\sigma}_{11} = \lambda_1^{-1/4} - \lambda_1^{-3/2}$. The achieved finite element results for both special cases have very good agreement with the achieved results by the analytical expressions, see figure 3.

Simple shear deformation has the form of $\boldsymbol{F} = \boldsymbol{I} + \gamma\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_2$ where γ is the amount of shear that is related to angle of shear θ by $\gamma = \tan\theta$. The Cauchy stress for hyperelastic material (26) becomes $(1/\mu)\,\boldsymbol{\sigma} = \sigma_{22}\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \sigma_{12}\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)$ where $\sigma_{22} = -\gamma^2$ and $\sigma_{12} = \gamma$. In general for hyperelastic material, the normal stress components do not vanish but the Cauchy stress components satisfy the universal relation $\sigma_{11} - \sigma_{22} = \gamma\sigma_{12}$, this is known as Poynting effect [36]. The hyperelastic Cauchy stress $\sigma_{12} = \gamma$ is increasing monotonically with amount of shear as plotted in figure 4, this theoretically means that the cube can sustain infinite shear deformation. The internally balance shear stress component shows softer response for $0 < \beta \leq 1$. The achieved curves of normalized shear stress is ordered such as hyperelastic ($\beta = 0$) is the most stiff, it becomes softer with increasing β, and the result of $\beta = 1$ is the most soft response. The internally balanced curves tend to reach an asymptotic value at large deformation. This agrees with previous findings in [19, 35].

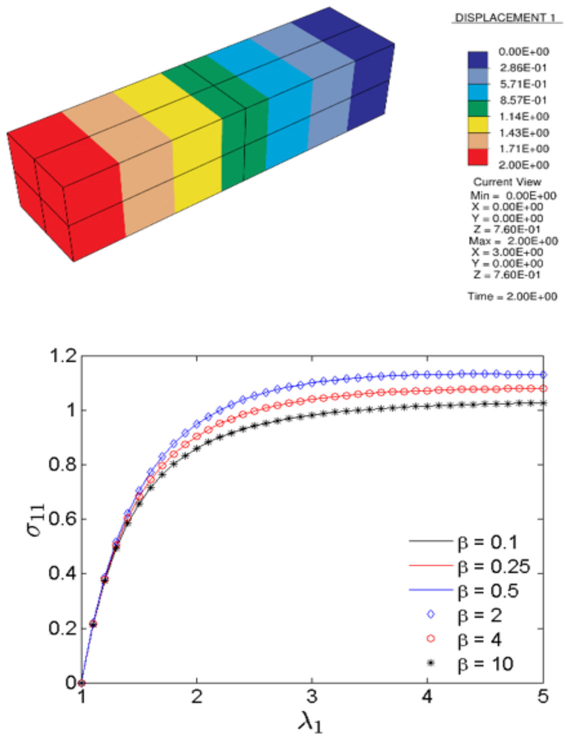

Figure 1. Uniaxial homogeneous deformation (top). Uniaxial Cauchy stress for different values of β (bottom). Achieved Cauchy stress value for given β coincides with achieved Cauchy stress value for $1/\beta$.

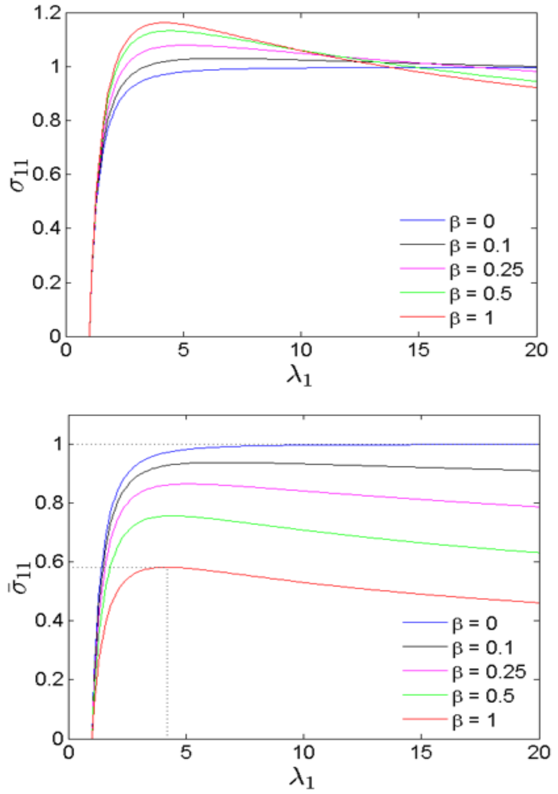

Figure 2. Uniaxial Cauchy stress for different values of β (top) and normalized Cauchy stress by $\mu(1+\beta)$ (bottm). The hyperelastic case is retrieved by $\beta = 0$.

9. Conclusion

In this work a finite element treatment is demonstrated for a material model based on new theory that applies argument of variation to both counterparts of deformation gradient multiplicative decomposition [21]. The use of Cayley – Hamilton equation facilitates significantly the implementation of internally balanced material model that has the form of $W(\hat{I}_2, \hat{I}_3, \check{I}_2, \check{I}_3)$. The response of the material model is examined in uniaxial loading and simple shear. The internally balanced theory retrieves the conventional hyperelastic theory in the special limiting case $\beta = 0$. The uniaxial stress for internally balanced material has a stiffer response compared with hyperelastic uniaxial stress up to significant value of stretching when it reaches maximum value then it shows softening behavior. For simple shear, the internally balanced shear stress shows softer response and reaches an asymptotic value in contrast with unbounded increases of hyperelastic shear stress. The presented treatment complements previous formulation demonstrated in [21] and it allows the use of complicated material models.

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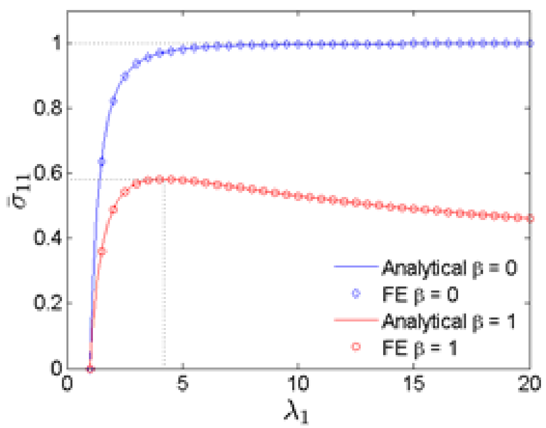

Figure 3. The verification of finite element results by analytical solution for two special cases $\beta = 0$ and $\beta = 1$.

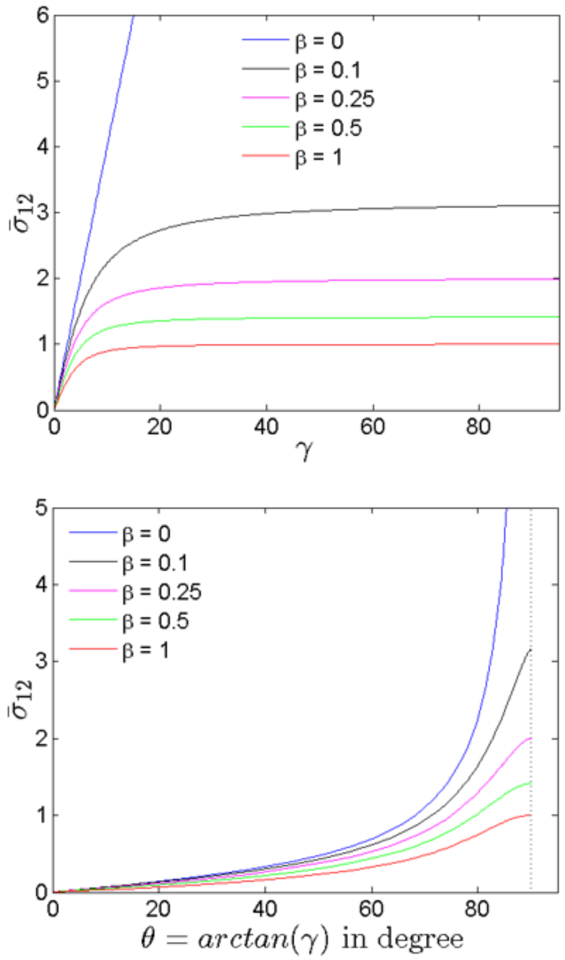

Figure 4. Normalized shear stress component $\bar{\sigma}_{12}$ for different values of β.

Appendix

Let $\boldsymbol{A}$ denote a second order symmetric tensor. The principal invariants of $\boldsymbol{A}$ are

$$I_1 = tr(\boldsymbol{A}), I_2 = \frac{1}{2}\left(\left(tr(\boldsymbol{A})\right)^2 - tr\left(\boldsymbol{A}^2\right)\right), I_3 = \det \boldsymbol{A} \quad (45)$$

and their partial derivatives with respect to $\boldsymbol{A}$

$$\frac{\partial I_1}{\partial \boldsymbol{A}} = \boldsymbol{I}, \frac{\partial I_2}{\partial \boldsymbol{A}} = I_1\boldsymbol{I} - \boldsymbol{A}, \frac{\partial I_3}{\partial \boldsymbol{A}} = I_3\boldsymbol{A}^{-1}. \quad (46)$$

Cayley – Hamilton equation can be written as

$$\boldsymbol{A}^3 - I_1\boldsymbol{A}^2 + I_2\boldsymbol{A} - I_3\boldsymbol{I} = \boldsymbol{0}, \quad (47)$$

successive multiplication of (47) by $\boldsymbol{A}^{-1}$ provides additional useful formulas such as

$$\boldsymbol{A}^2 - I_1\boldsymbol{A} + I_2\boldsymbol{I} - I_3\boldsymbol{A}^{-1} = \boldsymbol{0}, \quad (48a)$$

$$\boldsymbol{A} - I_1\boldsymbol{I} + I_2\boldsymbol{A}^{-1} - I_3\boldsymbol{A}^{-2} = \boldsymbol{0}. \quad (48b)$$

Derivative of $\boldsymbol{A}^{-1}$ and $\boldsymbol{A}^{-2}$ with respect to $\boldsymbol{A}$ are

$$-\frac{\partial \boldsymbol{A}^{-1}}{\partial \boldsymbol{A}} = \boldsymbol{A}^{-1}\odot\boldsymbol{A}^{-1}, \quad (49)$$

$$-\frac{\partial \boldsymbol{A}^{-2}}{\partial \boldsymbol{A}} = \boldsymbol{A}^{-2}\odot\boldsymbol{A}^{-1} + \boldsymbol{A}^{-1}\odot\boldsymbol{A}^{-2}, \quad (50)$$

where the $\odot$ operator is defined as

$$2\boldsymbol{A}^{-1}\odot\boldsymbol{A}^{-1}{}_{|IJKL} = A^{-1}_{IK}A^{-1}_{JL} + A^{-1}_{IL}A^{-1}_{JK}. \quad (51)$$

The push forward of the internal balance quantities is defined as

$$\boldsymbol{F}\breve{\boldsymbol{C}}^{-2}\boldsymbol{F}^T = \widehat{\boldsymbol{B}}\boldsymbol{B}^{-1}\widehat{\boldsymbol{B}}, \quad (52a)$$

$$\boldsymbol{F}\breve{\boldsymbol{C}}^{-1}\boldsymbol{F}^T = \widehat{\boldsymbol{B}}, \quad (52b)$$

$$\boldsymbol{F}\breve{\boldsymbol{C}}^{-1}\boldsymbol{C}\breve{\boldsymbol{C}}^{-1}\boldsymbol{F}^T = \widehat{\boldsymbol{B}}^2, \quad (52c)$$

$$\boldsymbol{F}\breve{\boldsymbol{C}}^{-1}\boldsymbol{C}^{-1}\boldsymbol{F}^T = \widehat{\boldsymbol{B}}\boldsymbol{B}^{-1}. \quad (52d)$$

The individual parts of (21) are

$$\widehat{\boldsymbol{\Xi}}_1 = \boldsymbol{F}\frac{\partial W}{\partial \hat{I}_1}\frac{\partial \hat{I}_1}{\partial \breve{\boldsymbol{C}}}\boldsymbol{F}^{\boldsymbol{T}} = -\frac{\partial W}{\partial \hat{I}_1}\widehat{\boldsymbol{B}}^2, \quad (53)$$

$$\widehat{\boldsymbol{\Xi}}_2 = \boldsymbol{F}\frac{\partial W}{\partial \hat{I}_2}\frac{\partial \hat{I}_2}{\partial \breve{\boldsymbol{C}}}\boldsymbol{F}^{\boldsymbol{T}} = \frac{\partial W}{\partial \hat{I}_2}\left(\widehat{\boldsymbol{B}}^3 - \hat{I}_1\widehat{\boldsymbol{B}}^2\right) =$$

$$\frac{\partial W}{\partial \hat{I}_2}\left(-\hat{I}_2\widehat{\boldsymbol{B}} + \hat{I}_3\boldsymbol{I}\right), \quad (54)$$

$$\widehat{\boldsymbol{\Xi}}_3 = \boldsymbol{F}\frac{\partial W}{\partial \hat{I}_3}\frac{\partial \hat{I}_3}{\partial \breve{\boldsymbol{C}}}\boldsymbol{F}^{\boldsymbol{T}} = -\frac{\partial W}{\partial \hat{I}_3}\hat{I}_3\widehat{\boldsymbol{B}}, \quad (55)$$

$$\breve{\boldsymbol{\Xi}}_1 = \boldsymbol{F}\frac{\partial W}{\partial \breve{I}_1}\frac{\partial \breve{I}_1}{\partial \breve{\boldsymbol{C}}}\boldsymbol{F}^{\boldsymbol{T}} = \frac{\partial W}{\partial \breve{I}_1}\boldsymbol{B}, \quad (56)$$

$$\breve{\boldsymbol{\Xi}}_2 = \boldsymbol{F}\frac{\partial W}{\partial \breve{I}_2}\frac{\partial \breve{I}_2}{\partial \breve{\boldsymbol{C}}}\boldsymbol{F}^{\boldsymbol{T}} = \frac{\partial W}{\partial \breve{I}_2}\left(\breve{I}_1\boldsymbol{B} - \boldsymbol{B}\widehat{\boldsymbol{B}}^{-1}\boldsymbol{B}\right), \quad (57)$$

$$\breve{\boldsymbol{\Xi}}_3 = \boldsymbol{F}\frac{\partial W}{\partial \breve{I}_3}\frac{\partial \breve{I}_3}{\partial \breve{\boldsymbol{C}}}\boldsymbol{F}^{\boldsymbol{T}} = \frac{\partial W}{\partial \breve{I}_3}\breve{I}_3\widehat{\boldsymbol{B}}. \quad (58)$$

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