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Nonlinear Natural Frequencies and Primary Resonance of Euler-Bernoulli Beam with Initial Deflection using Nonlocal Elasticity Theory

Ma'en S. Sari [a,*] , Ahmad Al-Qaisia [b]

[a] *Department of Mechanical and Maintenance Engineering, German Jordanian university , Amman 11180 , Jordan*

[b] *Department of Mechanical Engineering, The University of Jordan , Amman 11942, Jordan*

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Abstract

In the present work, we study the primary responses of Euler-Bernoulli beam with initial imperfection/rise. The nonlocal elasticity theory was used to derive the mathematical model to account for the scale effect of the considered beam. One type of beams was considered in the analysis; simply supported beam. The multi-mode approach was used to obtain the reduced nonlinear temporal equations of motion that contain quadratic and cubic nonlinear terms. The method of multiple-scales was applied to obtain approximate analytical solutions for the nonlinear natural frequencies in addition to the primary and resonance responses. The obtained results were presented over a selected range of physical parameters for the two types of beams such as; beam initial rise/imperfection, scale effect parameter and excitation level.

Keywords: *Geometric nonlinearity, nonlocal elasticity, beam rise, scale parameter, resonance*.

1. Introduction

Microelectromechanical systems and nanoelectromechanical systems have gained remarkable consideration due to their significant role in different engineering and modern technology fields, such as aerospace, communications, composites, electronics. These structures have more superior mechanical, electrical, and thermal properties as compared to other structures at the normal length scale. These properties make them ideal for the use in highly sensitive and high frequency devices for different applications [1].

In order to design a realistic model of a micro or a nanostructure and to well understand, optimize, and improve their performance, the small-scale effects and the atomic forces must be taken into consideration. In objects at the micro and nano scales, the dimensions, wavelengths, and sizes of these structures are no longer considered much larger than the characteristic dimensions of the microstructure. In these cases, the internal length scales of the material are comparable with the structure size. Moreover, the particles affect each other by long range cohesive forces in addition to the contact forces and heat diffusion. Consequently, the internal length scale should be considered as a material parameter, called nonlocal parameter, in the constitutive and governing equations and relations.

Although the experimental and atomistic simulations and models are both capable of showing the effects of the small-scale on the mechanical properties of the micro/nanostructures, these methods are expensive and restricted by computational capacity. It is well known that the local continuum theories for beams (Euler and Timoshenko) and plates (Kirchhoff and Mindlin) are scale free; therefore they are not able to capture the small scale effect on the mechanical, electrical, and thermal properties for very small beam and plate like structures. This makes them inadequate in describing the dynamical behavior for these structures [2]. In order to apply the continuum mechanics approach in the analysis of the micro and nanostructures, logical and reasonable modifications that take into consideration the scale effect, should be proposed. For this purpose, several theoretical models have been suggested. Among these, the strain gradient theory, the modified coupled stress theory, and the nonlocal elasticity theory which will be utilized in this article to analyze the free vibration problem of nonlocal annular and circular Mindlin plates.

The nonlocal elasticity theory was introduced by Eringen [3] accounts for the small-scale effects arising at the nanoscale level. He assumed that the stress at a point is a function of the strains at all points in the domain. Many researchers applied the nonlocal elasticity theory to study the free vibration, buckling, deflection, and dynamic problems of micro and nanostructures. For example, Reddy [4] obtained analytical solutions for the bending,

* Corresponding author e-mail: maen.sari@gju.edu.jo.

buckling, and vibration problems for simply supported Euler, Timoshenko, Reddy, and Levinson beams using Eringen's nonlocal theory. Murmu and Adhikari [5] studied the nonlocal transverse in-phase and out-of-phase vibrations of double nanobeam systems, in which explicit closed form expressions for natural frequencies were derived. Shakouri *et al*. [6] applied the Galerkin approach to study the free vibration problem of nonlocal Kirchhoff plates with different boundary conditions. It was shown that the nonlocal parameter and Poisson's ratio have significant effects on the vibration. Wang *et al*. [2] applied the Hamilton's principle, Eringen's nonlocal elasticity theory, and Timoshenko beam theory to analyze the free vibration problem of micro/nanobeams. Their study concluded that the effects of small scale, rotary inertia, and transverse shear deformation are important on the vibration behavior of short and stubby micro/nanobeams.

Moreover, Murmu and Adhikari [7] applied the differential quadrature method and the nonlocal elasticity theory to study the free vibration of a rotating carbon nanotube modeled as an Euler-Bernoulli beam. It was shown that the vibration is significantly influenced by the angular velocity, preload, and the nonlocal parameter. Lu *et al*. [8] derived the dispersion relation for a harmonic flexural wave propagation in an Euler-Bernoulli beam, as well as the frequency equations and modal shape functions of the beam with different boundary conditions based on Eringen's nonlocal elasticity theory. Murmu and Pradhan [9] implemented the nonlocal elasticity theory to study the vibration response of single graphene sheets embedded in an elastic medium modeled as Winkler and Pasternak foundations. The differential quadrature method was employed in their analysis to solve the fundamental natural frequencies of plates with clamped and simply supported edges.

In a similar manner, Murmu and Pradhan [1] applied the nonlocal elasticity theory to investigate the free vibration problem of nanoplates under uniaxially prestressed conditions. In their study the differential quadrature method was utilized to obtain the fundamental natural frequencies for simply supported and clamped nanoplates. Moreover, it was observed that buckling occurs at a smaller critical compressive load compared to the classical plate theory. Gürses *et al.* [10] studied the free vibration analysis of thin nano-sized annular sector plates, where Eringen's nonlocal elasticity theory was utilized to formulate the equation of motion. Additionally, the discrete singular convolution method was applied after transforming the irregular physical domain into a rectangular domain by using geometric coordinate transformation. It was shown that the effects of the nonlocal parameter are significant in the vibration analysis. Hashemi *et al.* [11] applied an exact analytical approach along with Eringen's theory to study the free vibration problem of thick circular and annular functionally graded Mindlin nanoplates with different combinations of boundary conditions. The effects of the plate radius, material properties which vary through the material according to a power-law distribution, and the nonlocal parameter on the natural frequencies were examined. In another study, Hashemi *et al.* [12] introduced potential functions and used the separation of variables method to obtain closed form solutions for nonlocal rectangular Mindlin plates with Levy-type boundary conditions. In their study, the effects of the nonlocal parameter, thickness to length ratio, and aspect ratio on the natural frequencies were investigated.

Ansari and Arash [13] applied the generalized differential quadrature method, Eringen's nonlocal elasticity theory, and the molecular dynamics simulations. Their purpose was to carry out the vibration analysis of single layered graphene sheets modeled as rectangular Mindlin plates, and to evaluate the appropriate values of the nonlocal parameter appropriate to each boundary condition. Duan and Wang [14] obtained exact solutions for the axisymmetric bending of micro and nano circular plates under general loading using a nonlocal plate theory. It was concluded that nonlocal parameter has a significant effect on the deflections, moments, and bending stiffness.

As stated by Ansari *et al*. [15], structures at the micro and nano scales are capable of undergoing large deformations within the elastic limit, which makes the nonlinear analysis obviously important. In their study, the homotopy perturbation method was applied to study the nonlinear vibrations of multiwalled carbon nanotubes embedded in an elastic medium. They showed that changing the material of the elastic medium has an influence on the vibration characteristics.

Fu *et al*. [16] investigated the nonlinear free vibration of embedded multiwalled carbon tubes using the incremental harmonic balanced method. It was shown that the surrounding elastic medium, van der Waals forces and aspect ratio of the multi-wall nanotubes have significant effects on the amplitude frequency response curves. It was noticed that in [15] and [16] the small scale effects were not taken into consideration; therefore, Ansari *et al*. [17] developed a nonlocal elastic beam model and adopted the incremental harmonic balance method to investigate the effects of the length scale, geometrical parameters, temperature rise and the elastic medium on the nonlinear frequency and displacement of embedded multiwalled carbon nanotubes. On the other hand, during the fabrication, manufacturing, assembling, and handling of such mechanical parts, structures with an initial deflection (slack) may be produced. As a result, this deflection will have an influence on the dynamics, vibration, and stability characteristics of the structure.

Al-Qaisia and Hamdan [18] presented an analytical study of nonlinear frequency veering of an elastic Euler-Bernoulli, hinged-hinged with one torsional spring at one end, resting on a Winkler elastic foundation and subjected to a static lateral load with an initial 1/4 sine shape rise due a constant differential edge settlement. A combined numerical-analytical procedure which accounts for the nonlinear interdependence between the lateral deflection and induced axial force due to mid-plane stretching was used to determine the beam static deflection. The assumed single mode approach was used to obtain the nonlinear temporal equation which contains quadratic and cubic nonlinear terms. The harmonic balance method was used to solve the nonlinear free vibration frequency about the static equilibrium deflection. The results of simulation indicate that the vibration amplitude, depending on location of the veering point, has a significant effect on the frequency loci behavior. Also, they extended these analyses to primary resonance response and its stability under vertical uniformly distributed excitation comprised of a large static part and a harmonically time varying part [19]. The obtained results indicate that the coefficients of the quadratic and cubic nonlinear terms, can vary widely depending on system parameters, and in particular, these coefficients can take positive and negative values, which affect the number of equilibrium positions, while the behavior of the system whether it is of hardening or softening type.

Lacarbonara *et al*. [20] studied the nonlinear response and stability of a hinged–hinged uniform moderately curved beam with a torsional spring at one end. It was shown that varying the initial rise of the beam has an effect on the one-to one auto parametric resonance and on the experienced Hopf and homoclinic bifurcations. Additionally, Ouakad and Younis [21] used a 2D nonlinear curved beam model which was derived by applying a multimode Galerkin approach to study the coupled in-plane and out-of-plane displacements of a carbon nano tube with curvature. They showed that the natural frequencies, mode shapes, mode crossings and mode veering are affected by the variation of the level of slackness and the DC load.

Mayoof and Hawwa [22] studied the nonlinear vibration of a clamped-clamped single wall carbon nanotube with waviness (deflection). The elastic continuum mechanics theory, along with Hamilton's principle, was applied to formulate the problem and derive the equation of motion which involved quadratic and cubic nonlinearities. The dynamics response of the system was investigated, and phase portrait, Poincaré section, and time history diagrams were generated. The results revealed that the nanotube underwent period-doubling bifurcations that were turned into chaos.

Garcia-Sanchez *et al*. [23] detected the bending-mode vibrations of multi and single-wall carbon nanotubes using a scanning force microscopy method. For multiwalled nanotubes, it was found that the resonance frequency is consistent with the elastic beam theory, whereas it is significantly reduced for single-wall nanotubes due to slack generated from fabrication processes. Üstünel et *al*. [24] studied the vibrations of nanotubes modeled as clamped-clamped suspended one-dimensional elastic systems with a slack and downward external forces; it was found that the frequencies are highly affected by the slack.

Al-Qaisia and Hamdan [25], extended the two previous studies [18, 19] on frequency veering by studying the effect of an initial geometric imperfection wavelength, amplitude and degree of localization on the in-plane nonlinear natural frequencies veering and mode localization of an elastic Euler-Bernoulli beam resting on a Winkler elastic foundation. Results were presented for the nonlinear natural frequencies of the first three modes of vibration, for a selected range of physical parameters like; torsional spring constant, elastic foundation stiffness and amplitude and wavelength of a localized and non-localized initial slack.

The present work extends the previous studies in [18, 19 and 25] on beam like-structures with initial imperfections to include the small scale effect on the primary and sub harmonic responses, using the nonlocal elasticity theory to derive the mathematical model.

The present study is organized as follows: First, the governing partial differential equation for the local beam is presented. Then, by applying Eringen's nonlocal theory, the partial equation of motion for the nonlocal beam is derived. Furthermore, the Galerkin method is applied to obtain the reduced order model, using the multi-mode approach. The method of multiple scales is utilized to determine the nonlinear natural frequencies and the frequency response curves for the primary resonance at selected values of parameters for simply supported beams.

Problem Formulation

Mathematical Model

The governing nonlinear equation of motion of moderately large amplitude vibration of an Euler-Bernoulli beam with initial deflection, and subjected to a harmonic force is given by:

$$m\ddot{\hat{w}} + EI\hat{w}^{iv} + \hat{c}\dot{\hat{w}} - \frac{EA}{L}\left(\hat{w}_0'' + \hat{w}''\right) \int_0^L \left(\frac{1}{2}\hat{w}'^2 + \hat{w}_0'\hat{w}'\right)d\hat{x} = F_0\cos\left(\hat{\Omega}\hat{t}\right) \quad (1)$$

where $\hat{w}$ is the transverse deflection, $\hat{w}_0$ initial deflection "initial rise", l is the beam length, *A* and *I* are beam's area and principal moment of inertia of the cross section *I* respectively, m is the mass per unit length, *E* is the Young's modulus of Elasticity, F_0 is the excitation level, $\hat{c}$ is the coefficient of damping, $\hat{t}$ is the time, and $\hat{\Omega}$ is the frequency of excitation. The prime denotes the derivative with respect to the spatial coordinate $\hat{x}$, while the dot denotes the derivative with respect to time $\hat{t}$.

The bending moment $M\left(\hat{x},\hat{t}\right)$ is given as

$$M\left(\hat{x},\hat{t}\right) = -EI\ \hat{w}''\left(\hat{x},\hat{t}\right) \quad (2)$$

In light of Eq. (2), $M''\left(\hat{x},\hat{t}\right)$ is given by

$$M''\left(\hat{x},\hat{t}\right) = m\,\ddot{\hat{w}}\left(\hat{x},\hat{t}\right) + \hat{c}\dot{\hat{w}}\left(\hat{x},\hat{t}\right) - \frac{EA}{L}\left(\hat{w}_0''\left(\hat{x}\right) + w''\left(\hat{x},\hat{t}\right)\right) \int_0^L \left(\frac{1}{2}\hat{w}'^2\left(\hat{x},\hat{t}\right) + \hat{w}_0'\left(\hat{x}\right)w'\left(\hat{x},\hat{t}\right)\right)d\hat{x} - F_0\cos\left(\hat{\Omega}\,\hat{t}\right) \quad (3)$$

Nonlocal Theory

In local elasticity theory, the stress at a reference point in a body depends on the strain at the same point. On the other hand, In the nonlocal elasticity theory pioneered by Eringen, the stress at a point in a linear, homogeneous, isotropic, and elastic domain is related to the stress field at all points in the domain. Eringen's theory is based on the atomic theory of lattice dynamics and experimental results on phonon scattering and dispersion [3, 10].

For nonlocal linear elastic solids, the stress tensor t_{ij} is defined as:

$$t_{ij} = \int_V \alpha\left(\left|x' - x\right|\right)\sigma_{ij}(x')\,dV(x') \tag{4}$$

where x is a reference point in the elastic domain, $\alpha(|x' - x|)$ is the non-local kernel attenuation function. It introduces the nonlocal effects at the reference point x produced by the local stress σ_{ij} at any point x', and $|x' - x|$ is the distance in Euclidean form.

Eringen introduced a linear differential operator ς, defined by $\varsigma = 1-(e_0 l)^2\nabla^2$, in which e_0 is a material constant estimated by experiments or other models and theories [3]. The nonlocal theory relations could result in approximate solutions to those obtained by atomic theory. The value of e_0 was taken to be 0.39 in Eringen's analysis. Moreover, the constant l represents the internal characteristic length which is of the same order of the external length.

According to Eringen's theory, the integral constitutive relation of Eq. (4) could be simplified and have the following form:

$$\left(1-(e_0 l)^2\nabla^2\right) t_{ij} = \sigma_{ij} \tag{5}$$

Due to its simple form, Eq. (5) has been extensively employed by many researchers in applying the nonlocal theory to study and analyze the vibration and mechanics of micro and nanostructures. According to Eringen’s nonlocal elasticity theory, the stresses at a point in the body not only depend on the strain at that point, but also on the strains at all other points of the body [3]. Thus, the nonlocal constitutive relation for the moment is given as:

$$M(\hat{x},\hat{t}) - (e_0 a)^2 M''(\hat{x},\hat{t}) = -EI\,\hat{w}''(\hat{x},\hat{t}) \tag{6}$$

where a is an internal characteristic length (e.g. lattice parameter, granular distance, and distance between C-C bonds) [24].

Inserting Eq. (3) into Eq. (6), we obtain:

$$M(\hat{x},\hat{t}) = (e_0 a)^2 \begin{bmatrix} m\hat{\ddot{w}} + \hat{c}\hat{\dot{w}} - \dfrac{EA}{L}\left(\hat{w}_0'' + \hat{w}''\right) \\ \displaystyle\int_0^L \left(\frac{1}{2}\hat{w}'^2 + \hat{w}_0'\hat{w}'\right)d\hat{x} \\ -F_0\cos(\hat{\Omega}\hat{t}) \end{bmatrix} - EI\hat{w}'' \tag{7}$$

Substituting Eq. (7) into Eq. (6), the equation of the transverse motion of the nonlocal Euler- Bernoulli beam with initial deflection can be written as:

$$\begin{aligned} & m\hat{\ddot{w}} - (e_0 a)^2 m\hat{\ddot{w}}'' + EI\hat{w}^{iv} + \hat{c}\hat{\dot{w}} - (e_0 a)^2\hat{c}\hat{\dot{w}}'' \\ & -\frac{EA}{L}\left(\hat{w}_0'' + \hat{w}''\right)\int_0^L\left(\frac{1}{2}\hat{w}'^2 + \hat{w}_0'\hat{w}'\right)d\hat{x} \\ & +(e_0 a)^2\frac{EA}{L}\left(\hat{w}_0^{iv} + \hat{w}^{iv}\right)\int_0^L\left(\frac{1}{2}\hat{w}'^2 + \hat{w}_0'\hat{w}'\right)d\hat{x} \\ & = F_0\cos(\hat{\Omega}\hat{t}) - (e_0 a)^2\left(F_0\cos(\hat{\Omega}\hat{t})\right)'' \end{aligned} \tag{8}$$

To simplify Eq. (8), the following non-dimensional variables and parameters with respect to the cross sectional area radius of gyration $r = \sqrt{I/A}$, are introduced:

$$x = \frac{\hat{x}}{l},\quad w = \frac{\hat{w}}{r},\quad w_0 = \frac{\hat{w}_0}{r},$$

$$t = \hat{t}\sqrt{\frac{EI}{ml^4}},\quad \Omega = \hat{\Omega}\sqrt{\frac{ml^4}{EI}},$$

$$c = \frac{\hat{c}l^2}{\sqrt{mEI}},\quad F = \frac{F_0 l^4}{rEI},$$

and $\mu^2 = \dfrac{(e_0\hat{a})^2}{l^2}$

After substituting the above parameters, Eq. (1) in its simplest form is given as:

$$\begin{aligned} & \frac{\partial^2 w}{\partial t^2} - \mu^2\frac{\partial^4 w}{\partial x^2\partial t^2} + \frac{\partial^4 w}{\partial x^4} + c\frac{\partial w}{\partial t} - c\mu^2\frac{\partial^3 w}{\partial x^2\partial t} \\ & -\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w_0}{\partial x^2}\right)\int_0^1\left(\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial w_0}{\partial x}\right)dx \\ & +\mu^2\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w_0}{\partial x^4}\right) \\ & \int_0^1\left(\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial w_0}{\partial x}\right)dx = F\cos(\Omega t) \end{aligned} \tag{9}$$

The nonlinear integral-partial differential equation (9) can be discretized using the Galerkin’s approach by assuming [26, 27]:

$$w(x,t) = \sum_{n=1}^{N}\phi_n(x)q(t) \tag{10}$$

where N is the number of retained modes, $\phi_n(x)$ are the mode shapes of the linear, undamped, and unforced beam, and $q_n(t)$ are the generalized coordinates. Substituting Eq. (10) into Eq. (9), multiplying by $\phi_n(x)$ and integrating over the beam’s span, yields the set of the nonlinear ordinary equations:

$$\begin{aligned} & \ddot{q}_m + \omega_m^2 q_m = -c\dot{q}_m + A_{mij}q_i q_j + \\ & B_{mijk}q_i q_j q_k + p_m\cos(\Omega t) \end{aligned}, \tag{11}$$

m=1, 2,…, N

where:

$$A_{mij}=\frac{1}{\int_0^1 \phi_m^2 dx-\mu^2\int_0^1 \phi_m''\phi_m dx}\left(\begin{array}{l}\left(\left(\int_0^1 \phi_i''\phi_m dx\left(\int_0^1 \phi_j' w_0' dx\right)\right)+\left(\int_0^1 \phi_i'\phi_j' dx\left(\frac{1}{2}\int_0^1 w_0''\phi_m dx\right)\right)\right.\\ \left.-\left(\mu^2\int_0^1 \phi_i^{iv}\phi_m dx\left(\int_0^1 \phi_j' w_0' dx\right)\right)-\left(\mu^2\int_0^1 \phi_i'\phi_j' dx\left(\frac{1}{2}\int_0^1 w_0^{iv}\phi_m dx\right)\right)\right)\end{array}\right) \tag{12}$$

$$B_{mijk}=\frac{1}{\int_0^1 \phi_m^2 dx-\mu^2\int_0^1 \phi_m''\phi_m dx}\left(\left(\left(\int_0^1 \phi_i''\phi_m dx\left(\int_0^1 \frac{1}{2}\phi_j'\phi_k' dx\right)\right)-\left(\mu^2\left(\int_0^1 \phi_i^{iv}\phi_m dx\left(\int_0^1 \frac{1}{2}\phi_j'\phi_k' dx\right)\right)\right)\right)\right) \tag{13}$$

$$\omega_m^2=\frac{1}{\int_0^1 \phi_m^2 dx-\mu^2\int_0^1 \phi_m''\phi_m dx}\left(\int_0^1 \phi_m^{iv}\phi_m dx-\left(\left(\int_0^1 w_0''\phi_m dx\right)+\mu^2\left(\int_0^1 w_0^{iv}\phi_m dx\right)\right)\left(\int_0^1 w_0'\phi_m' dx\right)\right) \tag{14}$$

$$f_m=F\int_0^1 \phi_m dx$$

$$p_m=\frac{f_m}{\int_0^1 \phi_m^2 dx-\mu^2\int_0^1 \phi_m''\phi_m dx} \tag{15}$$

where f_m is the projection of the force F onto the m^{th} mode.

The initial deflection for the simply supported beam is given as:

$$w_0(x,t)=q_0\sin(\pi x) \tag{16}$$

It is worth mentioning that the initial deflection has a value of q_0 at the midsection of the beam, rather than it satisfies the boundary conditions.

The mode shapes for a simply supported beam are given as:

$$\phi_n(x)=\sin(n\pi x) \tag{17}$$

The boundary conditions for the simply supported beam are given as:

$$w(0,t)=\frac{\partial^2 w(0,t)}{\partial x^2}=w(L,t)=\frac{\partial^2 w(L,t)}{\partial x^2}=0 \tag{18}$$

The boundary conditions for the non-local and local (classical) beams are the same when both ends of the beam are fixed as the scale effect gets nullified. In the present study, the dynamic behavior of beams under consideration will be analyzed for primary resonance using the first five modes.

Perturbation Analysis

Free Vibrations

In the present study, the non-dimensional nonlinear natural frequencies of the nonlinear beam given in Eq. (4) can be obtained by using the Method of Multiple Scales (MMS). The second order approximation of the free un-damped nonlinear natural frequency ω_{NL} is expressed as a function of the vibration amplitude a and the parameters of the nonlocal simply supported beams, and given by the expression:

$$\omega_{NL}=\omega_n\left(1+\frac{\alpha_{\text{eff}}}{\omega_n}a^2\right) \tag{19}$$

where α_{eff} is the effective nonlinearity to be defined later, and a is the amplitude (the initial condition for the displacement) of the beam.

Primary Resonance

In case the beam is subjected to primary resonance of the *nth* mode, we assume that the contribution of the nth mode is of lower order that the contributions of other modes. Therefore, we assume $q_n(t)$ and $q_m(t)$ to be in the form of [28]:

$$\begin{aligned}q_n(T_0,T_2,\varepsilon)&=\varepsilon q_{n1}+\varepsilon^2 q_{n2}+\varepsilon^3 q_{n3}\\ q_m(T_0,T_2,\varepsilon)&=\varepsilon^2 q_{m2}+\varepsilon^3 q_{m3},\qquad m\neq n\end{aligned} \tag{20}$$

where in these equations, ε is considered as a small and dimensionless parameter, and the two time scales T_0 and T_2 are introduced as $T_0=\varepsilon^0 t$ and $T_2=\varepsilon^2 t$. In order to apply the multiple scales method, the effects of the damping and the excitation terms are scaled to balance the effect of the nonlinearity. Hence, c and p_n are scaled as $\varepsilon^2 c$ and $\varepsilon^3 p_n$. Applying the method of multiple scales yields the *frequency-response curve* as:

$$\sigma = \alpha_{eff} a_n^2 \pm \sqrt{\frac{p_n^2}{4\omega_n^2 a_n^2} - \frac{c^2}{4}} \tag{21}$$

where σ is a detuning parameter introduced such that

$$\Omega = \omega_n + \varepsilon^2 \sigma \tag{22}$$

and α_{eff} is the effective nonlinearity given by:

$$\alpha_{eff} = -\frac{1}{8\omega_n} \left(\sum_{j=1}^{N} A_{jnn} \left(A_{nnj} + A_{njn} \right) \left(\frac{2}{\omega_j^2} + \frac{1}{\omega_j^2 - 4\omega_n^2} \right) + 3B_{nnnn} \right)$$

Eq. (22) shows the nearness of the excitation frequency to the nth natural frequency,

Eq. (23) is substituted into Eq. (19) to obtain the nonlinear natural frequencies [28].

In this section, the excitation frequency Ω is very close to the *nth* linear frequency ω_n. Hence, other modes not being directly or indirectly excited will decay to zero with time due to the presence of damping.

The details of the MMS are omitted for brevity, and interested readers can refer to Emam's thesis [28].

Results and Discussion

The multi-mode approach (assumed mode method) is applied in the present article, where five modes were retained to obtain the reduced order model. In this section, dynamic behavior and the characteristics of the nonlocal beams considered herein were analyzed by examining the effective nonlinearity of the simply supported beam given in Eq. (23). The value of α_{eff} as a function of the beam rise q_0 and the scale parameter μ was calculated and presented in 3-D surface plots as shown in Figures (1-2). These Figures show that the value of α_{eff} may increase or decrease depending on the combination of the parameters q_0 and μ. It is known that the behavior of the nonlinear beam is either of hardening (the frequency increases with the amplitude) or softening type (the frequency decreases with the amplitude), depending on the value of the parameter α_{eff}. It is known that for $\alpha_{eff} > 0$, the nonlinear beam given in Eq. (11) exhibits a hardening type behavior and a softening type otherwise. Moreover, when the value of $\alpha_{eff} = 0$, the effects of the quadratic and cubic nonlinearities cancel each other and consequently the response resonance curve resembles that of the corresponding linear beam, which implies that the frequency of the beam does not depend on the amplitude.

The variation of the fundamental nonlinear frequency with the amplitude of the first mode of a simply supported SS beam with $q_0 = 0.2$ at different values of the dimensionless nonlocal parameter μ is presented in Figure 2. It is observed that the nonlinear frequency decreases by increasing the nonlocal parameter due to the decrease in the stiffness of the beam. In Eringen nonlocal elasticity theory, it may be viewed that atoms are bonded by elastic springs with finite value, while the classical local model assumes that the stiffness of springs have a value of infinity [2, 10]. Further, as $\alpha_{eff} > 0$ at $q_0 = 0.2$, the frequency increases by increasing the vibration amplitude. Thus, the behavior of the SS beam is of hardening type regardless the value of scale factor μ.

The variation of the fundamental nonlinear frequency with the amplitude of the first mode of the SS beam with $\mu = 0.4$ at different values of the beam rise parameter q_0 is presented in Figure 3. The Figure reveals that the behavior of the beam switches from hardening to softening as the value of the beam rise q_0 increases. It is worthwhile to mention that at a value of the initial rise between 0.4 and 0.6, the frequency is constant and independent of the amplitude (initial conditions). In this case, the beam exhibits a linear behavior. As the beam rise is further increased, the beam has a softening effect as $\alpha_{eff} < 0$. It can be seen that at a given value of the scale effect, the beam may have hardening, softening, or linear behavior depending on the value of the initial rise q_0. This observation may be useful in the design and analysis of industrial applications in which the dynamics of the micro/ nano beams are main part of them.

The frequency response of the primary resonance of the nonlinear system given in Eq. (27), describing the forced vibration of the beam systems, were analyzed and presented for selected values of the beam rise q_0 and the scale parameter μ. In Figure 4, the frequency response curves are generated for the simply supported beam with $c = 0.05,\ f = 0.5$, and $q_0 = 0.1$ at different values of the scale parameter μ. It is observed that as the scale parameter μ increases, the beam's behavior is switched from hardening to softening type. It is clear that the curves are bent to the right (hardening behavior); whereas at μ =0.8, the curve is bent to the left (softening behavior), and the hardening non- linearity at μ =0.1 is stronger than that at μ =0.5. Furthermore, as the scale parameter μ increases, the steady state amplitude of the first mode of the beam decreases.

In a similar manner, the frequency response curves of primary resonance are generated and presented in Figures 5 and 6 for the simply supported beam at $c = 0.05,\ f = 0.5$, and $\mu = 0.2$ at different values of the beam rise q_0. It is shown that the beam exhibits the hardening and softening type as the value of q_0 is increased. From Figure 5, it is observed that if the beam exhibits a hardening type, the steady state amplitude of the first mode of the beam increases when the initial rise q_0 increases. On the other hand, Figure 6 shows that if the beam exhibits a softening type, the steady state amplitude of the first mode of the beam decreases when the initial rise q_0 increases. In these curves, the stable and unstable branches can be observed in addition to the jump phenomenon.

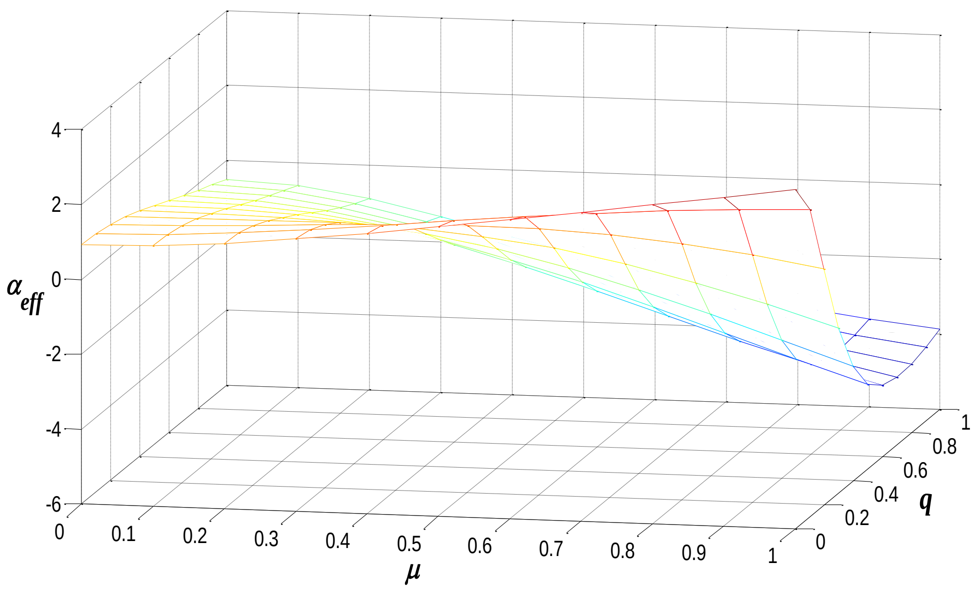

Figure 1: the parameter α_{eff} of the SS beam versus rise q and scale μ

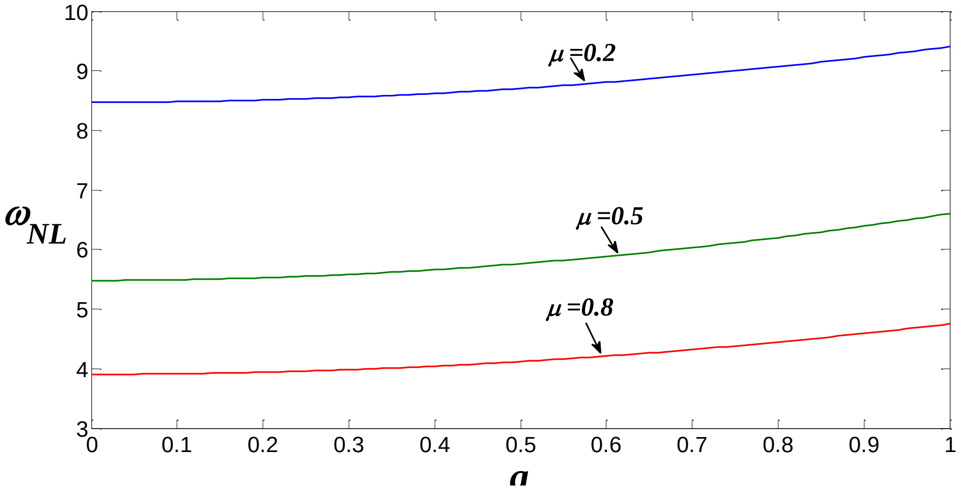

Figure 2: Variation of the fundamental nonlinear frequency of a simply supported beam with amplitude for $q = 0.2$

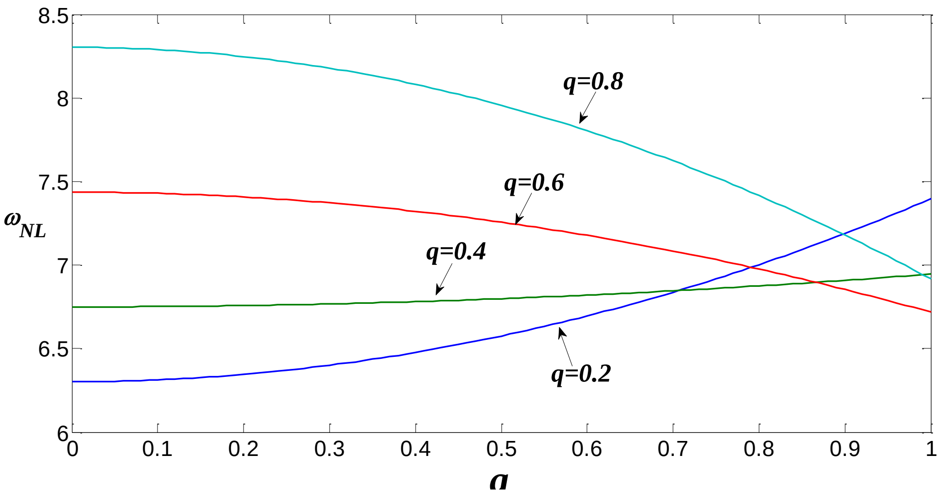

Figure 3: Variation of the fundamental nonlinear frequency of a simply supported beam with amplitude for $\mu = 0.4$

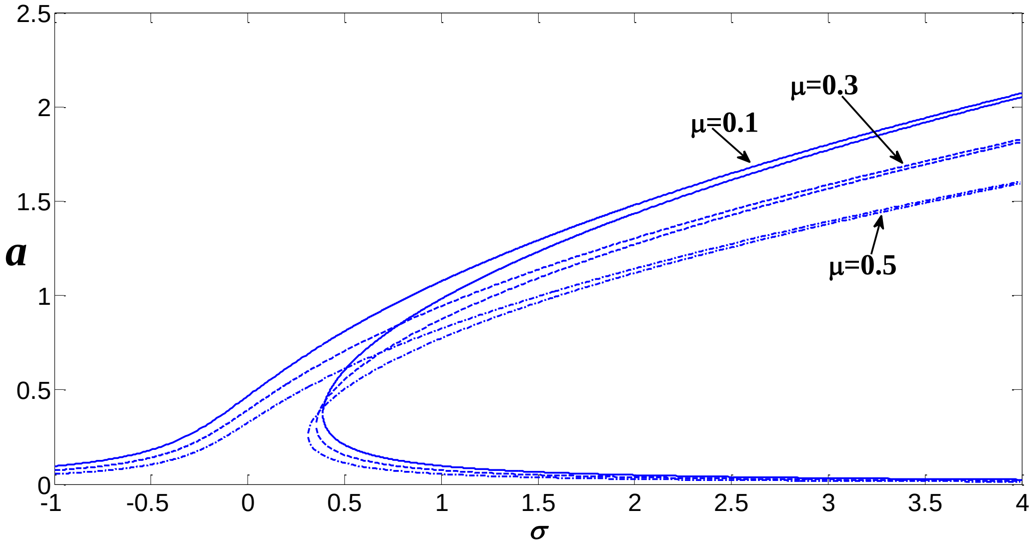

Figure 4: Frequency-response curves in the case of primary resonance of a simply supported beam for

$$c = 0.05,\ f = 0.5, q_0 = 0.1$$

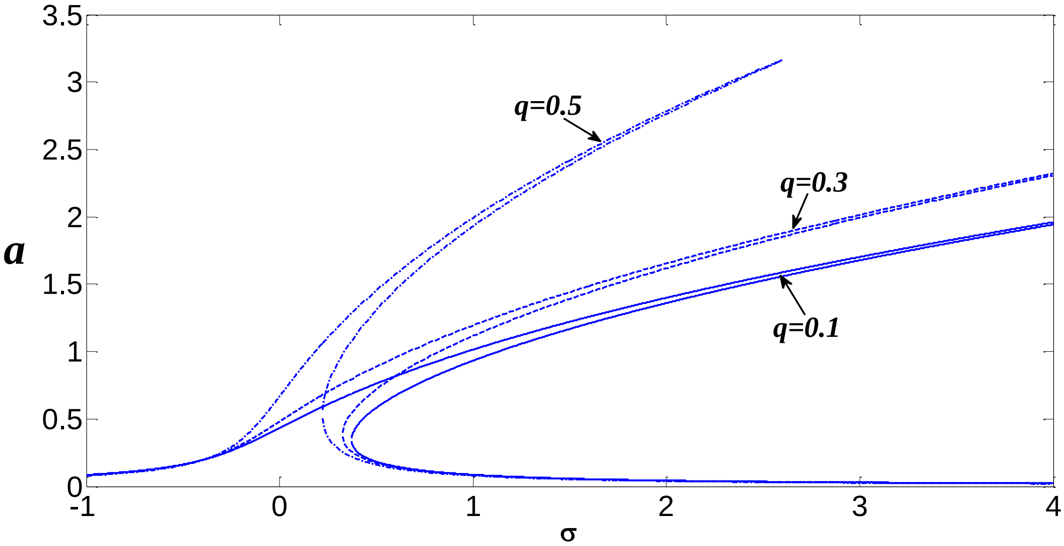

Figure 5: Frequency-response curves in the case of primary resonance of a simply supported beam for

$$c = 0.05,\ f = 0.5, \mu = 0.2$$

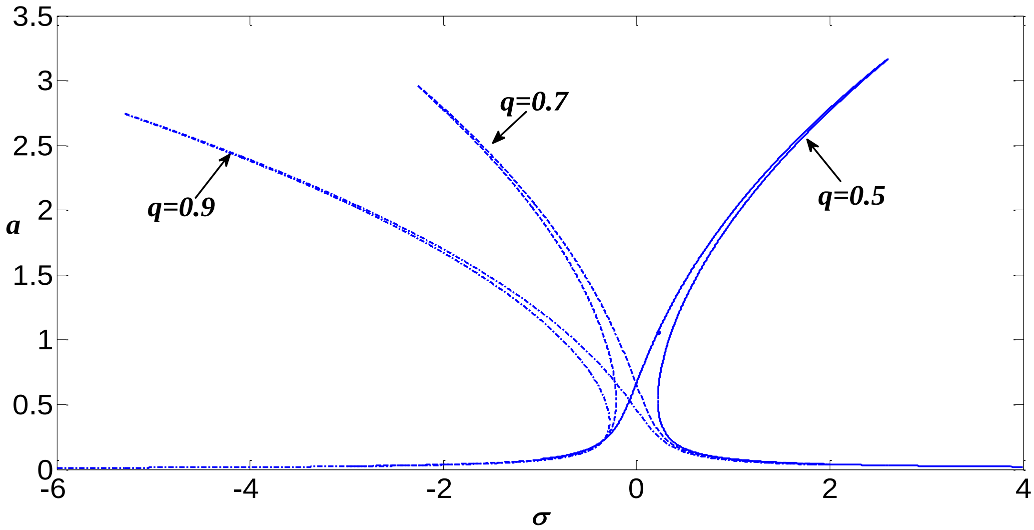

Figure 6: Frequency-response curves in the case of primary resonance of a simply supported beam for

$$c = 0.05,\ f = 0.5, \mu = 0.2$$

Conclusions

The primary resonance of a simply supported Euler beam with initial deflection was investigated. Eringen's nonlocal elasticity theory was utilized and the Galerkin approach was applied to convert the partial differential governing equation into a set of nonlinear ordinary differential equations. The method of multiple scales was carried out to determine the frequency response curves of the beam under consideration. It was shown that the scale parameter, beam's initial deflection, and excitation level have significant influence on the behavior of the beam. For the selected values of the scale parameter and the beam rise, it was shown that the simply supported beam switches its behavior from hardening to softening type.

For future work, it is recommended to consider other effects that may influence the behavior of micro and nano dynamical systems, such as temperature changes, electro and magnetic fields effects. Furthermore, analyzing such systems subjected to simultaneous resonances may be of high interest since these systems may exhibit mixed hardening and softening behavior.

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