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Stress Intensity Factors for Crack Located at an Arbitrary Position in Rotating FGM Disks

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Abstract

This article focuses on the stress analysis of an internal crack located at an arbitrary position in a rotating functionally graded material disk. The disk is assumed to be isotropic with exponentially varying elastic modulus in the radial direction.

A comprehensive study is carried out for various combinations of the crack length, direction, and location with the different gradation of materials. The results show that the material gradation, the crack position and the crack length have a significant influence on the value of stress intensity factors. Numerical results are given to assess the safety of the FGM and homogeneous cracked disks.

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Keywords: Functionally Graded Materials; Rotating Discs; Stress Intensity Factor; Stress Analysis; Crack Parameters.

1. Introduction

To meet the demands of new technologies, functionally graded materials (FGMs) are designed as systems that combine multiple and often significantly different materials into a single system.

FGMs are multiphase materials in which the volume fractions of the constituents vary continuously as a function of position. Therefore, the mismatch of thermo mechanical properties near the bond line is minimized. Another application area of FGMs includes their use as interfacial zone between two different layers; they are also used to improve the bonding strength [1], and to reduce the residual stresses, interfacial delamination [2] and stress concentration or stress intensity factors (SIFs)[3,4]. Because of their outstanding advantages over conventional composites and monolithic materials, these materials have received a wide attention of engineers and researchers from different fields of interest. Kim and Paulino [5] have addressed a wide variety of FGMs applications. The research in FGM development requires the supports of engineering mechanics, especially in the field of fracture mechanics. Recently a wide variety of researches are focused on analysis of cracked FGM structures. For example, Nami and Eskandari [6,7] considered cracked FGM cylinders and solved them in different conditions of loading.

Rotating discs are very common and useful parts of several engineering high speed rotating equipment such as compressors, cutters and grinding tools that are used extensively in the process industry today. Parallel to new industrial developments, it seems that the use of conventional materials in rotating discs is inadequate.

With the trend toward the analyses of cracked structural parts under centrifugal loading, much attention is being paid to investigate the strength and life of a rotating cracked disc. Unfortunately, the most studies are focused to homogeneous discs. For example, Tweed and Rooke [8] considered the homogeneous rotating disc with an edge crack, and Isida [9] considered it for a crack in an arbitrary position. The problem of three dimensional investigation of stress intensity factor in a cracked rotating impeller is considered by Nami and Eskandari [10]. A rigorous elastodynamic hybrid displacement finite element procedure for a safety analysis of fast rotating discs with mixed mode cracks is considered by Chen and Lin [11]. Cho and Park [12] have investigated the thermoelastic characteristics of functionally graded lathe cutting tools. Zenkour [13] considered a rotating FGM sandwich solid disk with material gradient in the thickness direction for the analysis of stress and displacement.

The problem of finite element analysis of thermoelastic field in a thin circular FGM disk with an exponential variation of material properties in radial direction is considered by Afsar and Go [14]. Sharma et al. [15] studied the thermoelastic displacements, stresses, and strains in a thin, circular, FGM disk subjected to thermal load by taking into account an inertia force due to rotation of the disk. Zafarmand and Hassani [16] obtained the elasticity solution of two-dimensional FGM rotating annular and solid disks with variable thickness.

In the present paper, an internal crack at an arbitrary location of a thin hollow circular FGM disk is considered (Fig. 1). A comprehensive study is carried out for various combinations of the crack length (2a), direction, and location with the different gradation of materials. Here the eccentricity of the midpoint of the crack line is measured

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as e and the orientation of the crack line relative to xdirection is shown as β .

2. Finite Element Formulation

Consider a rotating FGM disk with a concentric circular hole containing an internal crack at an arbitrary position as shown in Fig. 1. The FGM disk is considered to be made of two distinct material phases, which are, respectively, represented by the dark and white colors as shown in the figure. The distribution of each material continuously varies along the radial direction. The radii of the hole and outer surface of the disk are designated by \mathbf{R}_{in} and \mathbf{R}_{out} , respectively. Further, the angular

velocity of the disk is denoted by 60.

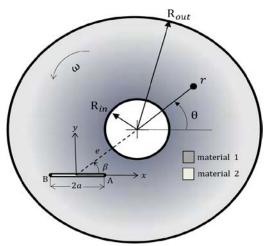


Figure 1. Rotating FGM disk containing an internal crack

A finite element code can be used to account for spatial variation in material property of FGM disk. There are different ways of incorporating changes in material properties into a finite element program. Walter *et al.* [17] describe two commonly-used methods. An element base method where the desired spatial material property of each element based on its location is achieved through a finite element code. Another way is to compute the material property at each integration point for element stiffness matrix via the spatial varying in material property function.

In this study, a finite element code is used to account for the material property changes in each element via its location. This section describes the details of the finite element formulation for stress and fracture analyses of FGM disk. The material is assumed to be isotropic with exponentially varying elastic modulus in radial direction as follows:

$$E(r) = E_{in} e^{\psi (r - R_{in})} \tag{1}$$

Where \mathbf{R}_{int} is the inner radius of the disk and $\boldsymbol{\psi}$ is the constants of material which defined as

$$\psi = \frac{1}{R_{in} - R_{out}} \ln(\frac{E_{in}}{E_{out}})$$
⁽²⁾

Which \mathbf{R}_{in} and \mathbf{R}_{out} denote the inner and outer radius of the disk, E_{in} and E_{out} are the values of elastic modulus at the inner and outer radius of the disk, respectively. For a simple traceable solution, the dependency to the Poisson's ratio is neglected and it is assumed constant ($\nu = 0.3$) throughout the disk.

The stress intensity factor for the disk is considered in the non-dimensional form and is defined as:

In which $\mathbf{K}_{\mathbf{I},\mathbf{A}}$ and $\mathbf{K}_{\mathbf{I},\mathbf{B}}$ are the calculated values of the first mode stress intensity factors in right and left sides of the crack, respectively. $\mathbf{K}_{\mathbf{II},\mathbf{A}}$ and $\mathbf{K}_{\mathbf{II},\mathbf{B}}$ are the calculated values of the second mode stress intensity factors in right and left sides of the crack, respectively.

The nominal stress intensity factor, $\mathbf{K}_{\mathbf{0}}$, for FGM disk is used as:

$$K_0 = \sigma_0 \sqrt{\pi a}$$
⁽⁴⁾

Where σ_0 is considered as:

$$\sigma_0 = \frac{3+\nu}{8} \rho V^2 \tag{5}$$

 \boldsymbol{a} is the crack length, $\boldsymbol{\rho}$ and \boldsymbol{V} being the material density and the peripheral speed, respectively.

The eccentricity of the crack center is defined as:

$$\varepsilon = \frac{e}{R_{out}}$$
(6)

and the dimensionless crack length is defined as:

$$\lambda = \frac{u}{R_{out} - e} \tag{7}$$

The FGM disk, considered in the present study, is assumed to be fixed to a shaft. The outer surface of the disk is free from any mechanical load. Thus, the boundary condition of the problem can be given by:

(i)
$$r = \mathbf{R}_{in}$$
, $u_r = \mathbf{0}$. (8)

(i) $r = R_{out}$, $\sigma_r = 0$.

3. Crack Tip Fields in FGMs

Material non-homogeneity has a significant influence on SIFs, which, in turn, will influence subsequent crack trajectory [18]. Williams [19] proposed the Eigen function expansion technique to investigate the nature of the neartip fields in a two-dimensional crack body. An extension of this conventional procedure has been used by Eischen [20] to establish the general form of the stress and displacement fields near a crack tip in a nonhomogeneous material with a spatially varying material property. Eischen solved the problem for materials with continuous, bounded, and differentiable property variations. He showed that the asymptotic fields for a crack in an FGM with continuous mechanical properties are similar to those of a crack embedded in a homogeneous material. In addition, the asymptotic displacement expressions for the homogeneous materials can be used for FGMs on condition that the material properties are calculated at the crack-front location. Jin and Noda [21] further showed that this result is also valid for materials with piecewise differentiable property variation.

A crack in a continuously non-homogeneous, isotropic and linear elastic FGM body with applied boundary conditions on the body satisfies the equilibrium equation:

$$\sigma_{ii,i} + F_i = 0 \tag{9}$$

and the Hooke's law is as (10):

$$\sigma_{ij} = C_{ijkl}(X)\varepsilon_{kl}$$
⁽¹⁰⁾

Where σ_{ij} is the stress tensor, F_i is the body force tensor, $C_{ijkl}(X)$ is the constitutive relation of FGMs and ε_{kl} is the strain tensor. The comma after a quantity denotes the partial derivatives with respect to spatial variable.

As mentioned previously, for a simple traceable solution, the functional dependence to the Poisson's ratio is neglected and it is assumed constant throughout the analysis, thus Eq. (10) can be expressed as [19]:

$$\sigma_{ij} = \frac{E(X)}{2(1+\nu)} C^{H}_{ijkl} \varepsilon_{kl} = \frac{E(X)}{2(1+\nu)} C^{H}_{ijkl} u_{k,l}$$
⁽¹¹⁾

Where u_i is displacement component and C_{ijkl}^H is the constitutive relation of corresponding homogeneous material and can be written as [22]:

$$C_{ijkl}^{H} = \frac{2\nu}{1-2\nu} \, \delta_{ij} \, \delta_{kl} + \, \delta_{ki} \, \delta_{lj} + \, \delta_{kj} \, \delta_{li} \quad ^{(12)}$$

here δ_{ii} denotes the Kronecker delta tensor.

Since the nature of the stress singularity for continuously non-homogenous, isotropic and linear elastic solid is precisely the same as the well-known form applicable to homogeneous materials, irrespective of the particular form of the Young's modulus variation (Eischen,1987), the stress intensity factors can be obtained from crack-opening-displacements (CODs) as [22]:

$$\begin{cases} K_{\rm I} \\ K_{\rm II} \\ K_{\rm III} \end{cases} = \frac{\mu_{tip} \sqrt{2\pi}}{4(1-\nu)} \lim_{\delta \to 0} \frac{1}{\sqrt{\delta}} \begin{cases} \Delta u_{\zeta}(\delta) \\ \Delta u_{\xi}(\delta) \\ (1-\nu)\Delta u_{\eta}(\delta) \end{cases}$$
(13)

where $\mathbf{K}_{\mathbf{I}}$, $\mathbf{K}_{\mathbf{II}}$ and $\mathbf{K}_{\mathbf{III}}$ are opening, sliding and tearing modes of SIFs, $\boldsymbol{\mu}_{tip}$ is the shear modulus at the crack front, $\boldsymbol{\delta}$ which approaches zero is a small distance between specified node at crack-surface and a node at crack-front, and

 $\Delta u_I(X) = [u_I(X \in \text{upper crack surface}) - u_I(X \in \text{lower crack surface})]$

in which $I = \zeta$, ξ and η are the CODs in the local coordinate systems.

4. The Validation of the Method

4.1.1. Stresses in a Functionally Graded Strip

To justify the reliability of the FGM model, the semiinfinite functionally graded strip under the uniform tensile load (in the y-direction) has been considered in Fig. (2-a). Young's modulus is an exponential function of z, i.e.

 $E(z) = E_1 e^{\ln(\frac{E_2}{E_1})\frac{z}{W}}$, while Poisson's ratio is constant.

Figure (2-b) shows the normalized σ_{yy} stresses for different levels of material gradation. These results agree well with those of Erdogan and Wu [23]. Thus, such excellent results validate the present FEM implementation for elastic FGMs.

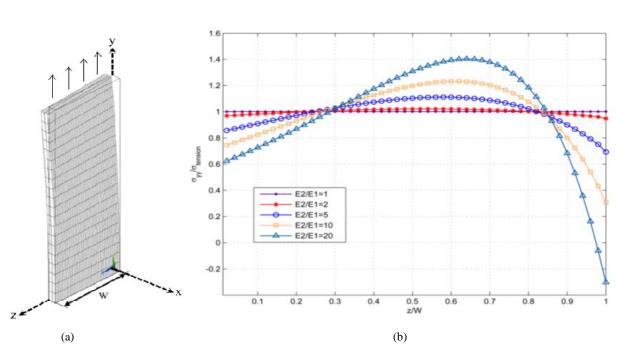


Figure 2. (a) Finite-element model of uncracked functionally graded strip, (b) normalized σ_{vvv} stresses for different levels of material gradation – uniform tensile far-field stress

(13)

4.1.2. Edge Crack in a Plate

Figure 3 shows an edge crack of length *a* located in a finite two-dimensional functionally graded strip under tension loading. As in the case of stress analysis of uncracked strip, the same form of material property gradation is considered. Table 1 compares the normalized SIFs of the current study with those reported by Chen *et al.* [21]. As can be seen in this table, the maximum difference is less than 2.25% when a/w=0.3 and $E_2/E=10$. Thus, good agreement is obtained between the different solutions for both homogeneous and FGM cases.

 Table 1. Normalized stress intensity factors for an edge cracked
 FGM plate under tension.

		a/W						
Method	E_2/E_1	0.2	0.3	0.4	0.5	0.6		
Chen et	0.2	1.455	1.897	2.529	3.443	4.926		
al.[24]	1	1.408	1.698	2.178	2.933	4.237		
	5	1.158	1.392	1.794	2.446	3.611		
	10	1.032	1.249	1.614	2.223	3.337		
Current	0.2	1.455	1.915	2.539	3.442	4.880		
study	1	1.428	1.733	2.205	2.951	4.216		
	5	1.182	1.430	1.826	2.472	3.603		
	10	1.046	1.282	1.658	2.273	3.357		

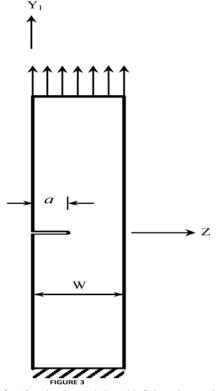


Figure 3. A functionally graded semi-infinite strip containing an edge crack

4.1.3. Arbitrary Crack in a Rotating Solid Disk

Consider the homogeneous elastic rotating disk in figure 1 with no hole. The stress intensity factors for the right hand side of the crack at different positions of the solid disk are determined and compared with those reported by Isida [9]. The problem was solved by two different methods, i.e. the Displacement Correlation Technique (DCT) and the J-integral method. The results are shown in figures 4. As it can be seen from figure 4, the results agree well with those reported in literature by Isida [9].

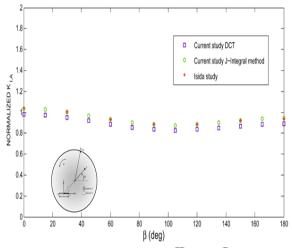


Figure 4. Variation of the normalized $K_{I,A}$ with β for $\lambda = \varepsilon = 0.5$

5. Results and Discussion

In this study, the stress intensity factors for two sides of an internal crack located at an arbitrary position of a rotating FGM disk with a concentric circular hole is determined. When the gradation of material is assumed radially, i.e. Eq. (1), it is obvious that $K_{I,A}(\beta) = K_{I,B}(180 - \beta)$ for $90 \le \beta \le 180$. Therefore, the problem for both $K_{I,A}$ and $K_{I,B}$ in the range of $0 \le \beta \le 90$ is solved and the results are plotted only for SIF in the right hand side of the crack. Figure 5 shows the variation of normalized $K_{I,A}$ with dimensionless crack lengths, λ_s for different gradation of materials, i.e. $\frac{E_{in}}{E_{out}} = 0.2, 1, 5$ and 20, when $\beta = 0, 30, 90, 120, 150$ and 180.

From Fig. 5, it is evident that for a certain value of λ_{s} in the range of $0 \le \beta \le 120$, higher the gradation of materials, i.e. $\frac{E_{in}}{E_{out}}$, higher the $K_{I,A}$. An exception is seen only for radial cracks with $\lambda = 0.1$ and $\frac{E_{in}}{E_{out}} = 20$. Variation of the normalized stress intensity factor for

Variation of the normalized stress intensity factor for right hand side of the crack, i.e. $K_{I,A^{\mu}}$ with β is plotted at figure 6a through 6d. Each curve is plotted for a certain value of material gradation and different values of dimensionless crack length, *i.e.*, $\lambda = 0.1$ to 0.5. As seen, for homogeneous materials, the larger the crack lengths

with $\frac{E_{in}}{E_{out}} < 1$ and $\beta \leq 40$, the smaller the crack lengths (larger $\lambda' s$), the higher the values of $K_{I,A}$ and this rule tends to be reversed for higher values of $\beta' s$. This fact can be seen in figure 5, too.

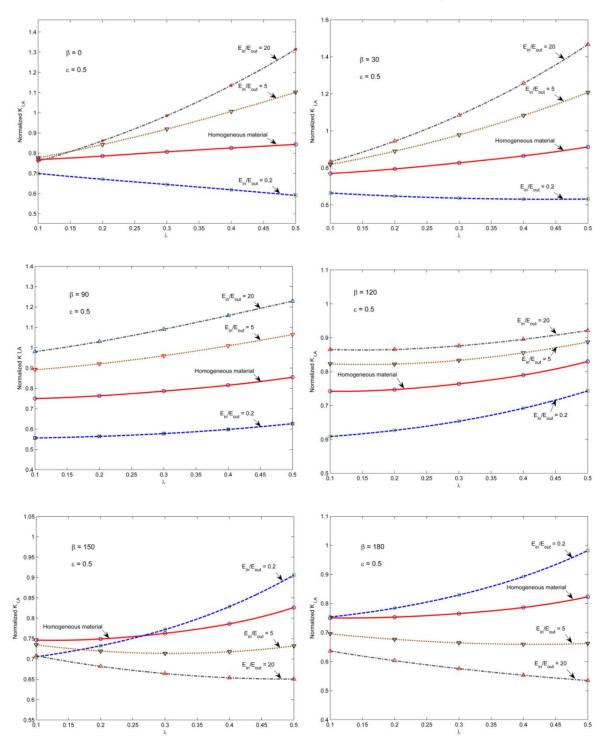




Figure 5. Variation of normalized $K_{I,A}$ with λ for different gradation of materials, i.e. $\frac{\omega_{in}}{r} = 0.2, 1, 5$ and 20, when $\beta = 0, 30, 90, 120, 150$ and 180.

Another interesting point for homogenous and FGM disks with $\frac{E_{in}}{E_{out}} > 1$, is that $K_{I,A}$ for the crack tip which closer to the disk center is generally larger than $K_{I,B}$. In other words, for FGM disks with $\frac{E_{in}}{E_{out}} < 1$, $K_{I,A}$ is smaller than $K_{I,B}$. In addition, it can be concluded from Figs. 6c-6d that as $\frac{E_{in}}{E_{out}}$

increased, there exists a value of β in which the $K_{I,A}$ is independent to the crack length. Here $\beta_1 \approx 132^\circ$. This position for FGM disks with $\frac{E_{In}}{E_{out}} < 1$ occurs at $\beta_2 \approx 180^\circ - 132^\circ = 48^\circ$. It seems that these values are supplement.

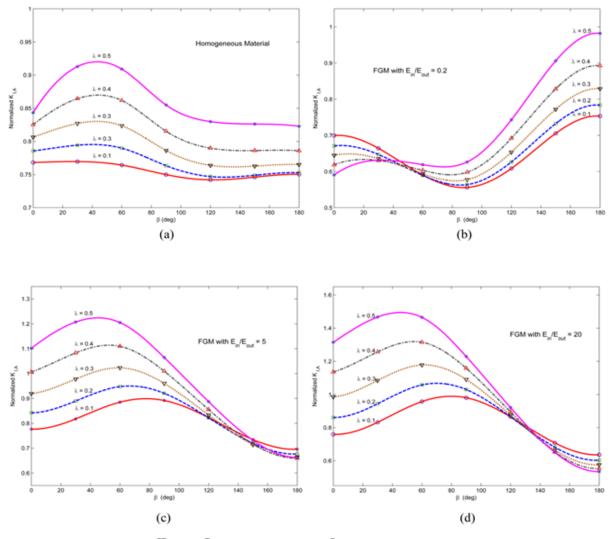


Figure 6. Variation of normalized $K_{I,A}$ with β for different crack lengths, λ_s and different gradation of materials, (a) homogenous material, (b) FGM with $\frac{E_{in}}{E_{out}} = 2$, (c) FGM with $\frac{E_{in}}{E_{out}} = 5$, (d) FGM with $\frac{E_{in}}{E_{out}} = 20$

The values of $K_{I,A}$ in homogeneous and functionally graded discs for small and large cracks are plotted at Figs. 7a and 7b. It can be seen from Figures 7a – 7b that for cracks in functionally graded discs with $\frac{E_{in}}{E_{out}} > 1$, maximum $K_{I,A}$ occurs at $0 < \beta < 180$, where the cracks are not in radial direction. Also, the minimum value of SIF occurs at left hand side of the crack in the position of $\beta = 0$ (radial crack).

In addition, it can be concluded from Fig. 6a that small cracks in FGM disks with $\frac{E_{in}}{E_{out}} < 1$, have minimum value of SIFs at $0 < \beta < 180$ and maximum values

occur in radial cracks. For a wide variety of crack positions in the range of $0 \le \beta \le 90$, large cracks in functionally graded discs with $\frac{E_{in}}{E_{out}} < 1$, experience constant value of SIFs.

So far, the stress intensity factors K_{I} for the opening mode deformation have been discussed. On the other hand, the shear mode stress intensity factor K_{II} are found to be rather small compared with K_{I} of the corresponding crack tips. This fact may be attributed to the biaxial tension state of the untracked disk. Table 2 gives $K_{II,B}$ values for the typical case when $\beta = 60$, and they are shown to be fairly small compared with the corresponding $K_{I,B}$ values

given in figure 5. Therefore, the shear mode deformation can be neglected in the fracture analysis of a rotating disk.

Table 2. Values of $K_{II,B}$ for $\beta = 60$ and $\varepsilon = 0.5$

	a						
E_{in}/E_{out}	0.1	0.2	0.3	0.4	0.5		
0.2	0.0720	0.0728	0.0786	0.0903	0.1085		
1	0.0092	0.0159	0.0255	0.0390	0.0571		
5	0.0474	0.0311	0.0140	0.0050	0.0267		
20	0.0847	0.0587	0.0337	0.0084	0.0185		

Variation of normalized $K_{I,A}$ with ε for different gradation of materials in cracks with constant length at $\beta = 45^{\circ}$ is shown in figure 8. As seen, for cracks with $\varepsilon < 0.7$, higher the material gradation, higher the values of $K_{I,A}$. In other words, for cracks near the outer radius of the disk with $\varepsilon > 0.7$, the values of $K_{I,A}$. In other words, the values of $K_{I,A}$ decrease with increasing the gradation of material. Here the value of $\varepsilon \approx 0.7$ can be considered as an important point in fracture analysis of rotating disks.

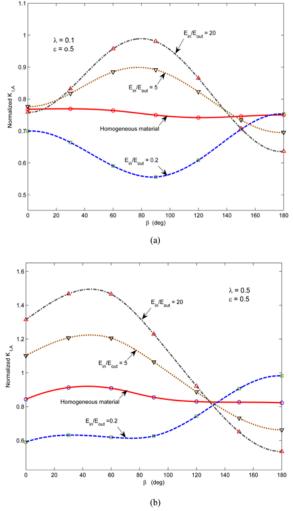


Figure 7. Comparison of $K_{I,A}$ in homogeneous and functionally graded materials for, (a) relatively small cracks ($\lambda = 0.1$), (b) relatively large cracks ($\lambda = 0.5$).

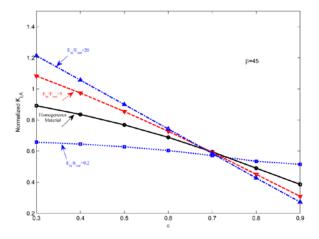


Figure 8. Variation of normalized $K_{I,A}$ with ε for different gradation of materials, i.e. $\frac{\omega_m}{\omega} = 0.2, 1, 5$ and 20, in cracks with constant length at $\beta = 45$

6. Summary and Conclusions

In this study, the stress analysis of an internal crack located at an arbitrary position in a thin hallow rotating FGM disk is carried out. The disk is assumed to be isotropic with exponentially varying elastic modulus in the radial direction. A comprehensive study is carried out for various combinations of the crack length, direction, and location with the different gradation of materials. The results which are normalized for the advantage of nondimensional analysis show that the material gradation, the crack position and the crack length have a significant influence on the amount of stress intensity factors. The critical values of stress intensity factors and their position in homogeneous and FGM disks are obtained. The larger the cracks, the larger the stress intensity factors in homogeneous disks. In general, this is not valid in FGM disks. Numerical results are given to assess the safety of the FGM and homogeneous cracked disks.

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