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Studies On $\overline{X}$ - Control Chart With Pareto In-Control Times for Non Normal Variates

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Abstract:

In this paper we develop and analyse economic statistical design of $\overline{X}$ control chart with the assumption that the sample average of the quality characteristic (follows a Johnson distribution and the process in-control times follow Pareto distribution. The Johnson distribution is generally taken for all types of skewed and kurtic variables. Here, the Pareto distribution is chosen since in many production processes at places like Fertilisers, chemicals, etc., the in-control times are having long upper tail and suits to the Pareto distribution. The expected cost per a unit time is derived with the use of the cost model developed for the $\overline{X}$ control chart. Minimizing the expected cost per a unit time, the optimal design parameters like sample size and the time interval between two successive samples are derived for given Type I and Type II errors associated with the control chart. The sensitivity of the model with respect to the parameters and costs are also studied. This design is extended to the case when the time to search for an assignable cause and time to repair are also random and follow a Weibull distribution. The effect of randomness on these times is also investigated.

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Keywords*:* $\overline{X}$ Control Chart;In-Control Times;Out Of Control Times;Johnson Distribution;Expected Cost Per Unit Time.

Notations:

S = Expected number of samples taken during the in-control state.
ARL_0= Average run length when process is in control
ARL_1= Average run length when process is out of control
h = Time interval between successive samples
k = Number of standard deviations from control limits to centre line
Δ = Number of standard deviations slip when out of control
n = Sample size
E = Expected sampling time for one observation
δ_1= Indicator variable to indicate whether production continues or not during the assignable
cause search, δ_1=1 if production continues and δ_1=0, otherwise
δ_2= Indicator variable to indicate whether production continues or not during the repair process, δ_2=1 if production continues and δ_2=0, otherwise
T_0= Expected assignable cause search time for a false alarm
T_1= Expected time to identify the assignable cause
T_2 = Expected time to repair the process
a = Fixed cost per sample
b = Variable cost per sample
C_0 = Hourly cost due to nonconformities produced while the process is in control
C_1 = Hourly cost due to nonconformities produced while the process is out of control (C_1> C_0)
C_2 = Cost per false alarm
W = Cost for locating and repairing the assignable cause
α = Probability that $\overline{X}$ falls outside the control limits when the process is in control
β= Probability that $\overline{X}$ falls within the control limits when the process is out of control
E(C) = Expected Cycle cost
E(T) = Expected Cycle time

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1. Introduction:

For the excellence in productivity, one has to concentrate on the quality improvement programs. The major issue of achieving excellence in quality depends on Quality control. One of the important techniques adopted for process control is $\overline{X}$ control chart. With the help of Central limit theorem, many researchers considered that the sample mean of the quality characteristic follows normal distribution[1]. However, this assumption is suitable only when the sample size is large. But in many practical situations, the sample size of the quality character is small and the normal assumption leads to error. Taking this concept into consideration several researchers developed $\overline{X}$ control charts and statistical economic design with various distributions [2-6].

Recently Huifen Chen and Yuyen Cheng [7] have considered the economic statistical design of $\overline{X}$ chart with the assumption that the sample average, $\overline{X}$ of the quality character follows Johnson distribution. Their cost model was based on the model proposed by Mc Williams [8], which is an extension of the work of Lorenzen and Vance [9]. They emphasised the need of utilising Johnson distribution as $\overline{X}$ distribution. They have also assumed that the in-control times of the process follow a Weibull distribution. One of the major drawbacks of the two parameter Weibull distribution is that it considers the failure starts from zeroth time. In many practical situations once the process is put in control it may take a minimum period to failure. Hence it is reasonable to consider a distribution for the process in-control times which characterize this property. One such distribution often used in reliability and life testing is Pareto distribution.

The Pareto distribution also characterizes a limiting distribution of the waiting time (time to exceed a specific value of the process character). This distribution is named after an Italian, Vilfredo Pareto (1848-1923). It is also empirically observed that in Chemical industries, the in-control times of the process are having left – skewed with long upper tail depicting the frequency distribution of Pareto. Very little work has been reported in literature regarding the Economic design of $\overline{X}$ control chart with Pareto in-control times even though this distribution is quite common in many Manufacturing and Production processes. Hence in this article , we develop and analyze the Economic statistical design of the $\overline{X}$ control chart with the assumption that the quality characteristic $\overline{X}$ follows Johnson distribution with mean 'μ' and variance 'σ^2' and the in control times of the process are random and follows a Pareto distribution with probability density function of the form,

$$f(t) = \{c.\theta^c.t^{-(c+1)}\}, (\theta > 0,\ c > 0,\ t \geq \theta) \quad (1)$$

The Pareto Cumulative distribution function is

$$F(T) = \left\{1 - \left(\frac{\theta}{t}\right)^c\right\}, (\theta > 0,\ c > 0,\ t \geq \theta) \quad (2)$$

where, 't' is the in-control time, 'θ' is the parameter of Pareto distribution, 'c' is the shape parameter and its mean is $\left(\frac{c.\theta}{c-1}\right)$ [10]. The various shapes of the Pareto distribution frequency curves for different values of the parameter, 'c' are shown in Figure1.

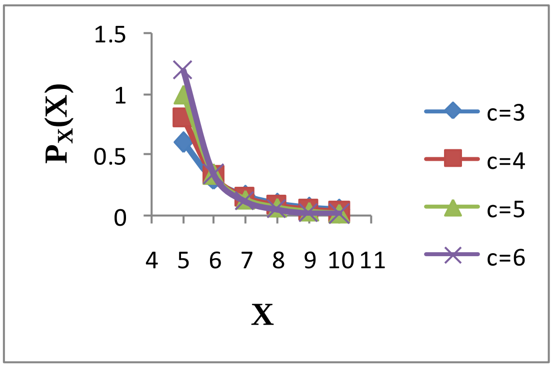

Figure 1. Pareto Distribution Frequency curves (Θ=5)

The expected hourly cost equals the ratio of the expected cycle cost to the expected cycle time. The schematic diagram of a Production cycle is shown in Figure2 as given by Lorenzen and Vance [9].

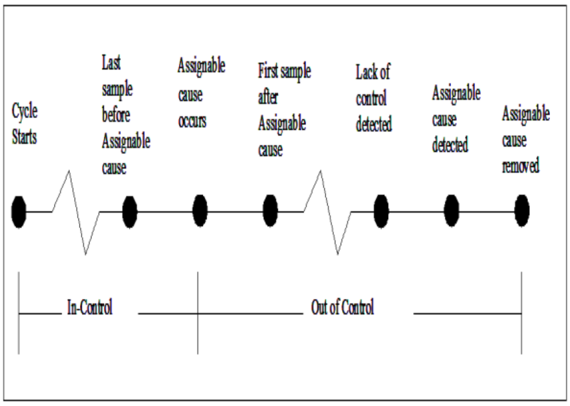

Figure 2. Production Cycle

The optimal design parameters of the $\overline{X}$ control chart namely, the Sample size (n) and the Sampling interval (h) are derived by minimizing the expected cost per unit time. The sensitivity of the model with respect to the parameters and costs is also studied. This model is extended to the case when the out of control times (the time to identify assignable causes and time to repair) are also random and follows Weibull distribution.

2. Cost Model

The production process is assumed to start in an in-control state. In order to detect a shift in the process mean, a sample of 'n' independent quality characteristic measurements $X_1, X_2, X_3 \dots X_n$ is taken at intervals of 'h' hours. The sample average $\overline{X}$ is assumed to have Johnson distribution. Johnson family, proposed by Normanl Johnson [10], includes three transformations of the standard normal distribution. Let Y and X denote the Johnson and standard normal variables, respectively. We use the transformation

$$X = \gamma + \delta \ln\left(\frac{Y-\xi}{\xi+\eta-Y}\right), 0 \leq (Y-\xi) \leq \eta, \quad (3)$$

The constants 'ξ' and 'η' are location and scale parameters, 'γ' and 'δ' are the shape parameters. To compute the Johnson cumulative probability F(y) = P{Y ≤ y}, we transform Y to X using Equation (3) and then let F(y) = Φ(x), where 'Φ' is the standard normal cumulative distribution function. Here, Y is taken as a bounded normal distribution. Hence,

$$F_\mu(y) = \Phi\left[\gamma + \delta \ln\left(\frac{Y-\xi}{\xi+\eta-Y}\right)\right] \tag{4}$$

where, $F_\mu(\,.\,)$ is the Johnson cumulative distribution function with mean, 'μ' and standard deviation, $\frac{\sigma}{\sqrt{n}}$.

For independent observations , average run lengths when the process is in control and out of control i.e., ARL_0 and ARL_1 respectively are related to Type I and Type II error probabilities, 'α' and 'β' as follows :

$$ARL_0 = 1 / \alpha$$

where,

$$\alpha = P\{\bar{X} < \mu_0 - k\sigma/\sqrt{n} \text{ or } \bar{X} > \mu_0 + k\sigma/\sqrt{n} \mid \mu = \mu_0\}$$

$$= 1 + F_{\mu_0}\left(\mu_0 - \frac{K\sigma}{\sqrt{n}}\right) - F_{\mu_0}\left(\mu_0 + \frac{K\sigma}{\sqrt{n}}\right) \tag{5}$$

and $ARL_1 = 1 / (1-\beta)$

where,

$$\beta = P\{\mu_0 - k\sigma/\sqrt{n} \le \bar{X} \le \mu_0 + k\sigma/\sqrt{n} \mid \mu = \mu_0 + \Delta\sigma\}$$

$$= F_{\mu_0+\Delta\sigma}\left(\mu_0 + \frac{K\sigma}{\sqrt{n}}\right) - F_{\mu_0+\Delta\sigma}\left(\mu_0 - \frac{K\sigma}{\sqrt{n}}\right) \tag{6}$$

The $\bar{X}$ control chart is designed to detect whether the process is out of control or not.

The design parameters 'n' and 'h' are chosen to minimize the expected cost per a unit time i.e., E(C)/E(T). A quality cycle is defined as the time until the next in - control period. The in-control times in each cycle are identically and independently distributed. Hence, the expected hourly cost E(C/T) equals the ratio of the expected cycle cost to the expected cycle time.

From Figure2, the expected cycle time consists of 4 parts namely, (1) Expected time elapsed before assignable cause occurs , (2) Expected time between the occurrence of the assignable cause and the next out of control signal, (3) Expected time 'T_1' to identify the assignable cause and (4) Expected time 'T_2' to repair the process. In this model it is assumed that the process in-control times follow Pareto distribution with mean, $\left(\frac{c.\theta}{c-1}\right)$ and the in-control times in each cycle are independently identically distributed with probability density function of the form given in Equation (1). Therefore, the expected time elapsed before the assignable cause occurs, when the production ceases during the search for an assignable cause, is the mean of the in-control times plus the time spent searching during false alarms.

The expected time spent during false alarms is 'T_0' times the expected number of false alarms

$$= \frac{T_0.S}{ARL_0} \tag{7}$$

where, 'T_0' is the expected search time for a false alarm,

'S' is the expected number of samples taken while in control and

'ARL_0' is the average run length while in control.

We have,

$$S = \sum_{i=0}^{\infty} i * pr(\text{assignable cause occurs between the } i^{th} \text{ and } (i+1)^{th} \text{samples})$$

$$= \sum_{i=0}^{\infty} i\left[\left(\frac{\theta}{i.h}\right)^c - \left(\frac{\theta}{(1+i).h}\right)^c\right] \tag{8}$$

where, $\left\{1 - \left(\frac{\theta}{t}\right)^c\right\}$ is the cumulative distribution function of Pareto distribution (in-control time) as given in Equation (2).

If the process is shut down during searches, the expected time equals to

$$\left(\frac{c.\theta}{c-1}\right) + \frac{T_0.S}{ARL_0}$$

Let δ_1=1 if production continues during searches and δ_1=0 if production ceases during searches.

Hence, the expected time until the assignable cause occurs is

$$\left(\frac{c.\theta}{c-1}\right) + \frac{[(1-\delta_1).T_0.S]}{ARL_0}$$

$$= \left(\frac{c.\theta}{c-1}\right) + (1-\delta_1).T_0.S.\alpha \tag{9}$$

The total number of samples taken is the sum of expected number of samples taken during the in control time (S), plus the number of samples when the process has gone out of control (ARL_1)

$$= (S + ARL_1)$$

The time interval between sampling is 'h'. Hence, the total time period for taking ($S+ARL_1$) samples is

$$= (S + ARL_1).h$$

But, the samples are being taken out every 'h' hours irrespective of whether the process is in or out of control. Here, we require only the time period between the occurrence of assignable cause and the next out of control signal which is simply the total time period minus the mean time of in control state.

It is assumed that the in control times follow Pareto distribution with mean$\left(\frac{c.\theta}{c-1}\right)$.

Hence, the required time is

$$(S + ARL_1).h - \left(\frac{c.\theta}{c-1}\right)$$

As 'E' is the expected time for measuring each observation, for a sample of 'n' items, the time to analyze the sample and chart the result is 'nE'. Hence the total expected time between the occurrence of assignable cause and the next out of control signal is

$$(S + ARL_1).h - \left(\frac{c.\theta}{c-1}\right) + n.E$$

$$= \left(S + \frac{1}{(1-\beta)}\right).h - \left(\frac{c.\theta}{c-1}\right) + n.E \tag{10}$$

As 'T_1' is the expected time to identify the assignable cause and 'T_2' is the expected time to repair the process, the total time required when the process is out of control is

$$\left(S + \frac{1}{(1-\beta)}\right).h - \left(\frac{c.\theta}{c-1}\right) + n.E + T_1 + T_2 \tag{11}$$

Therefore, from Equations (9) & (11) the expected cycle time is

$$E(T)=\left(\frac{c.\theta}{c-1}\right)+(1-\delta_1).T_0.S.\alpha+\left(S+\frac{1}{(1-\beta)}\right).h-\left(\frac{c.\theta}{c-1}\right)+n.E+T_1+T_2$$
$$=(1-\delta_1).T_0.S.\alpha+\left(S+\frac{1}{(1-\beta)}\right).h+n.E+T_1+T_2 \qquad (12)$$

The cost of the entire cycle includes (1) Cost of non conformities, (2) Cost of false alarms, (3) Expected cost for sampling and charting the result and (4) Cost of repairs, 'W'.

Let C_0 = Hourly cost due to nonconformities produced while the process is in control and

C_1 = Hourly cost due to nonconformities produced while the process is out of control ($C_1 > C_0$)

Assuming that the production continues during both search and repair, the expected cost per cycle due to non conformities

$$= C_0\,(\text{Mean in control time}) + C_1\,(\text{Total time required when the process is out of control})$$
$$= C_0\left(\frac{c.\theta}{c-1}\right)+C_1\left\{\left(S+\frac{1}{(1-\beta)}\right).h-\left(\frac{c.\theta}{c-1}\right)+n.E+T_1+T_2\right\}$$
$$= C_0\left(\frac{c.\theta}{c-1}\right)+C_1\left\{\left(S+\frac{1}{(1-\beta)}\right).h-\left(\frac{c.\theta}{c-1}\right)+n.E+\delta_1 T_1+\delta_2 T_2\right\} \qquad (13)$$

where, 'δ_1' and 'δ_2' are as defined earlier.

The expected number of false alarms $=\dfrac{S}{ARL_0}$

The expected cost of false alarms

$$= C_2.\left(\frac{S}{ARL_0}\right) = C_2.S.\alpha \qquad (14)$$

where, 'C_2' is the cost per false alarm.

Since the fixed cost per sample (a) and the variable cost per sample (b) are considered to effect the total cost, the expected cost for sampling and charting the result is given by (a+b.n) times the total time producing divided by the time interval between sampling

$$=\frac{(a+b.n)}{h}.\left\{\left(S+\frac{1}{(1-\beta)}\right).h+n.E+\delta_1.T_1+\delta_2.T_2\right\} \qquad (15)$$

From Equations (13), (14) and (15), we have,

The Expected cycle cost is,

$$E(C)=C_0\left(\frac{c\theta}{c-1}\right)+C_1\left\{\left(S+\frac{1}{(1-\beta)}\right).h-\left(\frac{c\theta}{c-1}\right)+n.E+\delta_1.T_1+\delta_2.T_2\right\}+C_2.S.\alpha+\frac{(a+b.n)}{h}.\left\{\left(S+\frac{1}{(1-\beta)}\right).h+n.E+\delta_1.T_1+\delta_2.T_2\right\}+W \qquad (16)$$

in the design of $\bar{X}$ chart , the design parameters 'n' and 'h' are chosen to minimize the expected

cost per hour 'Z' for a quality cycle where,

$Z=E(C)/E(T)$ (17)

Substituting the Equations (12) and (16) in Equation (17), we get,

$$Z=\left\{C_0\left(\frac{c\theta}{c-1}\right)+C_1\left\{\left(S+\frac{1}{(1-\beta)}\right).h-\left(\frac{c\theta}{c-1}\right)+n.E+\delta_1.T_1+\delta_2.T_2\right\}+C_2.S.\alpha+\frac{(a+b.n)}{h}.\left\{\left(S+\frac{1}{(1-\beta)}\right).h+n.E+\delta_1.T_1+\delta_2.T_2\right\}+W\right\}\Big/\left\{(1-\delta_1).T_0.S.\alpha+\left(S+\frac{1}{(1-\beta)}\right).h+n.E+T_1+T_2\right\} \qquad (18)$$

Where, 'α' and 'β' are as defined in Equations (5) and (6) respectively.

The optimum values for 'h' and 'n' are obtained by differentiating 'Z' with respect to 'h' and 'n' and equating them to zero.

Let

$$R=\frac{\partial s}{\partial h}=\frac{\partial}{\partial h}\left\{\sum_{i=0}^{\infty} i\left[\left(\frac{\theta}{i.h}\right)^c-\left(\frac{\theta}{(1+i).h}\right)^c\right]\right\}$$
$$=\sum_{i=0}^{\infty}\frac{\partial}{\partial h}\left\{i\left[\left(\frac{\theta}{i.h}\right)^c-\left(\frac{\theta}{(1+i).h}\right)^c\right]\right\}$$
$$=\sum_{i=0}^{\infty}\left[i.c.\theta^c.h^{-(c+1)}.\left(\frac{1}{(1+i)^c}-\frac{1}{i^c}\right)\right] \qquad (19)$$

we have,

$$\frac{\partial[E(C)]}{\partial h}=C_1.\left[S+\frac{1}{(1-\beta)}+h.R\right]+C_2.\alpha.R+\frac{(a+b.n)}{h}.\left\{S+\frac{1}{(1-\beta)}+h.R\right\}-\frac{(a+b.n)}{h^2}.\left\{\left(S+\frac{1}{(1-\beta)}\right).h+n.E+\delta_1.T_1+\delta_2.T_2\right\} \qquad (20)$$

$$\frac{\partial[E(T)]}{\partial h}=S+\frac{1}{(1-\beta)}+h.R+(1-\delta_1).T_0.S.\alpha.R \qquad (21)$$

Therefore, $\dfrac{\partial Z}{\partial h}=0$ implies,

$$\left\{E(T).\left\{C_1.\left[S+\frac{1}{(1-\beta)}+h.R\right]+C_2.\alpha.R+\frac{(a+b.n)}{h}.\left\{S+\frac{1}{(1-\beta)}+h.R\right\}-\frac{(a+b.n)}{h^2}.\left\{\left(S+\frac{1}{(1-\beta)}\right).h+n.E+\delta_1.T_1+\delta_2.T_2\right\}-E(C).\left\{S+\frac{1}{(1-\beta)}+h.R+(1-\delta_1).T_0.S.\alpha.R\right\}\right\}\Big/[E(T)]^2=0$$

This implies,

$$\left\{\left\{(1-\delta_1).T_0.S.\alpha+\left(S+\frac{1}{1-\beta}\right).h+n.E+T_1+T_2\right\}.\left\{C_1.\left[S+\frac{1}{(1-\beta)}+h.R\right]+C_2.\alpha.R+\frac{(a+b.n)}{h}.\left\{S+\frac{1}{(1-\beta)}+h.R\right\}-\frac{(a+b.n)}{h^2}.\left\{\left(S+\frac{1}{(1-\beta)}\right).h+n.E+\delta_1.T_1+\delta_2.T_2\right\}\right\}-\left\{C_0\left(\frac{c.\theta}{c-1}\right)+C_1\left\{\left(S+\frac{1}{(1-\beta)}\right).h-\left(\frac{c.\theta}{c-1}\right)+n.E+\delta_1.T_1+\delta_2.T_2\right\}+C_2.S.\alpha+\frac{(a+b.n)}{h}.\left\{\left(S+\frac{1}{(1-\beta)}\right).h+n.E+\delta_1.T_1+\delta_2.T_2\right\}+W\right\}.\left\{S+\frac{1}{(1-\beta)}+h.R+(1-\delta_1).T_0.S.\alpha.R\right\}\right\}\Big/\left\{(1-\delta_1).T_0.S.\alpha+\left(S+\frac{1}{(1-\beta)}\right).h+n.E+T_1+T_2\right\}^2=0 \qquad (22)$$

Where, 'α' and 'β' are as defined in Equations (5) and (6) respectively.

And for optimal 'n',

$$\frac{\partial[E(C)]}{\partial n}=C_1.E+\frac{(a+b.n)}{h}.E+\left(\frac{b}{h}\right).\left[\left(S+\frac{1}{(1-\beta)}\right).h+n.E+\delta_1.T_1+\delta_2.T_2\right] \qquad (23)$$

$$\frac{\partial[E(T)]}{\partial n}=E \qquad (24)$$

Therefore, $\dfrac{\partial Z}{\partial n}=0$ implies,

$$\left\{E(T).\left\{C_1.E+\frac{(a+b.n)}{h}.E+\left(\frac{b}{h}\right).\left[\left(S+\frac{1}{(1-\beta)}\right).h+n.E+\delta_1.T_1+\delta_2.T_2\right]\right\}-E(C).E\right\}\Big/[E(T)]^2=0$$

This implies,

$$\left\{\left\{(1-\delta_1).T_0.S.\alpha+\left(S+\frac{1}{(1-\beta)}\right).h+n.E+T_1+T_2\right\}.\left\{C_1.E+\frac{(a+b.n)}{h}.E+\left(\frac{b}{h}\right).\left\{\left(S+\frac{1}{(1-\beta)}\right).h+n.E+\delta_1.T_1+\delta_2.T_2\right\}\right\}-\left\{C_0\left(\frac{c.\theta}{c-1}\right)+C_1\left\{\left(S+\frac{1}{(1-\beta)}\right).h-\left(\frac{c.\theta}{c-1}\right)+n.E+\delta_1.T_1+\delta_2.T_2\right\}+C_2.S.\alpha+\frac{(a+b.n)}{h}.\left\{\left(S+\frac{1}{(1-\beta)}\right).h+n.E+\delta_1.T_1+\delta_2.T_2\right\}+W\right\}.E\right\}\Big/\left\{(1-\delta_1).T_0.S.\alpha+\left(S+\frac{1}{(1-\beta)}\right).h+n.E+T_1+T_2\right\}^2=0 \qquad (25)$$

Where, 'α' and 'β' are as defined in Equations (5) and (6) respectively.

Solving the Equations (22) and (25) iteratively using numerical method the optimal sampling interval (h^*) and the optimal sample size (n^*) can be obtained for the given values of the model parameters and cost parameters

3. Sensitivity Analysis:

The sensitivity of the cost model is studied with respect to all the cost parameters involved in the model. The initial parameters of the cost model are set as follows:

c =2, θ = 5, δ_1= δ_2=1, E= 0.01, a = 3, b = 0.01, T_0=1, T_1=2, T_2=1, C_2=5, W = 90,
C_0=10, C_1=20, ξ=0.05, η=2, γ=2, δ=1, μ=2.5, σ=1, k=3, Δ=0.5

Using the Equations (22) and (25) and the initial values of the parameters as given above, the optimal interval between the successive samples (h^*) and the optimal sample size (n^*) are obtained. Substituting these values in the total cost, 'Z', the optimal total cost Z^* is computed and all these values are presented inTable-1and Table-2.

Table 1. Optimal values of n, h and Z for various values of c, θ, ξ, η, γ and δ

C	θ	ξ	η	γ	δ	h*	n*	Z*
1.890	5	0.05	2	2	1	12.048	146	19.819
1.895	5	0.05	2	2	1	12.020	139	19.823
1.990	5	0.05	2	2	1	11.683	15	19.887
1.995	5	0.05	2	2	1	11.677	9	19.890
2	5.05	0.05	2	2	1	11.709	27	19.881
2	5.10	0.05	2	2	1	11.766	51	19.869
2	5.15	0.05	2	2	1	11.838	75	19.857
2	5.20	0.05	2	2	1	11.922	99	19.845
2	5	0.01	2	2	1	11.949	6	19.893
2	5	0.02	2	2	1	11.882	5	19.893
2	5	0.03	2	2	1	11.814	4	19.893
2	5	0.04	2	2	1	11.744	3	19.893
2	5	0.05	1.80	2	1	13.357	90	19.922
2	5	0.05	1.85	2	1	12.878	67	19.916
2	5	0.05	1.90	2	1	12.437	45	19.910
2	5	0.05	1.95	2	1	12.035	23	19.902
2	5	0.05	2	1.92	1	13.343	114	19.933
2	5	0.05	2	1.94	1	12.801	84	19.926
2	5	0.05	2	1.96	1	12.343	55	19.916
2	5	0.05	2	1.98	1	11.968	27	19.905
2	5	0.05	2	2	0.96	15.266	68	19.856
2	5	0.05	2	2	0.97	14.504	55	19.863
2	5	0.05	2	2	0.98	13.658	40	19.871
2	5	0.05	2	2	0.99	12.405	22	19.881

Table 2. Optimal values of n, h and Z for various values of C_0, C_1, C_2, W, a, b and Δ

C_0	C_1	C_2	W	a	b	Δ	h*	n*	Z*
9.80	20	5	90	3	0.01	0.5	11.562	51	19.866
9.85	20	5	90	3	0.01	0.5	11.582	39	19.873
9.90	20	5	90	3	0.01	0.5	11.606	27	19.880
9.95	20	5	90	3	0.01	0.5	11.636	15	19.887
10	20.05	5	90	3	0.01	0.5	11.636	15	19.937
10	20.10	5	90	3	0.01	0.5	11.606	27	19.980
10	20.15	5	90	3	0.01	0.5	11.582	39	20.023
10	20.20	5	90	3	0.01	0.5	11.562	51	20.066
10	20	10	90	3	0.01	0.5	13.446	28	19.919
10	20	15	90	3	0.01	0.5	14.933	44	19.934
10	20	20	90	3	0.01	0.5	16.243	56	19.944
10	20	25	90	3	0.01	0.5	17.426	66	19.951
10	20	5	87	3	0.01	0.5	11.535	74	19.853
10	20	5	88	3	0.01	0.5	11.562	51	19.866
10	20	5	89	3	0.01	0.5	11.606	27	19.880
10	20	5	90	3	0.01	0.5	11.672	3	19.894
10	20	5	90	2	0.01	0.5	12.079	95	19.895
10	20	5	90	2.25	0.01	0.5	11.977	72	19.895
10	20	5	90	2.5	0.01	0.5	11.875	49	19.894
10	20	5	90	2.75	0.01	0.5	11.773	26	19.894
10	20	5	90	3	0.001	0.5	11.702	13	19.893
10	20	5	90	3	0.003	0.5	11.675	6	19.893
10	20	5	90	3	0.005	0.5	11.669	4	19.893
10	20	5	90	3	0.007	0.5	11.668	3	19.893
10	20	5	90	3	0.01	0.75	12.192	33	19.892
10	20	5	90	3	0.01	1	12.616	60	19.893
10	20	5	90	3	0.01	1.25	13.042	85	19.895
10	20	5	90	3	0.01	1.5	13.495	109	19.897

From Table 1 , it is observed that as the shape parameter, 'c' increases , the optimal values of 'n' is decreasing, the optimal values of 'h' is decreasing and the expected cost per hour decreases for fixed values of the other parameters. If the parameter, 'ϴ' increases, the optimal values of 'n' are increasing, the optimal values of 'h' is increasing and the expected cost per hour decreases for fixed values of the other parameters. Regarding the Johnson distribution parameters, it is observed that as the parameter 'ξ' increases, the optimal values of 'n' is decreasing, the optimal values of 'h' is decreasing and the expected cost per hour remains constant for fixed values of the other parameters. When the parameter 'η' increases , the optimal values of 'n' is decreasing, the optimal values of 'h' is decreasing and the expected cost per hour decreases for fixed values of the other parameters. With respect to the parameter 'γ' , if it increases , the optimal values of 'n' is decreasing, the optimal values of 'h' is decreasing and the expected cost per hour decreases for fixed values of the other parameters. As the parameter 'δ' increases , the optimal values of 'n' is decreasing, the optimal values of 'h' is decreasing and the expected cost per hour increases for fixed values of the other parameters.

1The variation in optimal design parameters for various values of 'c' and 'ϴ' are shown in Figures 3 and 4 respectively.

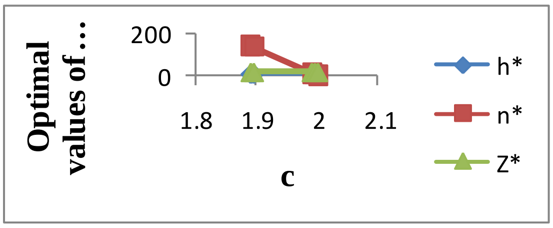

Figure 3. 'c' Vs Optimal values of h, n and Z

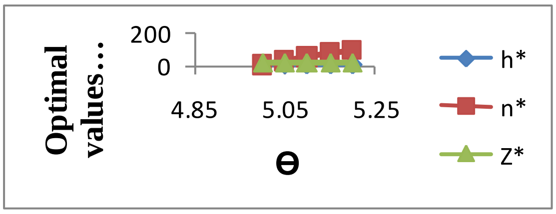

Figure 4. 'ϴ' Vs Optimal values of h, n, Z

From Table 2, it is observed that as the parameter 'C_0' increases, the optimal values of 'n' is decreasing, the optimal values of 'h' is increasing and the expected cost per hour increases for fixed values of the other parameters. When the parameter 'C_1' increases, the optimal values of 'n' is increasing, the optimal values of 'h' is decreasing and the expected cost per hour increases for fixed values of the other parameters .With regard to the parameter 'C_2', if it increases, the optimal values of 'n' is increasing, the optimal values of 'h' is increasing and the expected cost per hour increases for fixed values of the other parameters. As the parameter 'W' increases , the optimal values of 'n' is decreasing, the optimal values of 'h' is increasing and the expected cost per hour increases for fixed values of the other parameters. With reference to the parameter 'a', as it increases, the optimal values of 'n' is decreasing, the optimal values of 'h' are decreasing and the expected cost per hour decreases for fixed values of the other parameters. As the parameter 'b' increases, the optimal values of 'n' is decreasing, the optimal values of 'h' is decreasing and the expected cost per hour remains constant for fixed values of the other parameters. When the parameter 'Δ' increases , the optimal values of 'n' is increasing, the optimal values of 'h' is increasing and the expected cost per hour increases for fixed values of the other parameters

4. Optimal Design Parameters When the Process Out of Control Times are Random:

the earlier sections 3 and 4, we assumed that the time to identify the assignable causes for process out of control (T_1) and the time to repair or eliminate the assignable cause (T_2) are fixed and known. But in many production processes there are multiple assignable causes like defective raw materials, faulty setup, untrained operators, the cumulative effect of heat, vibration, shocks, power fluctuations, etc., when the process is governed by multiple assignable causes, 'T_1' and 'T_2' are also random and follows a probability distribution. A suitable distribution for 'T_1' and 'T_2' is a Weibull distribution since it accommodates constant, increasing and decreasing hazard rates. Hence, here we assume that 'T_1' and 'T_2' follow Weibull distributions with parameters (λ_1, ν_1) and (λ_2, ν_2) respectively, 'λ' being the scale parameter and 'ν' being the shape parameter.

$$f(t_1) = \lambda_1.\nu_1.t^{\nu_1-1}.e^{-\lambda_1.t_1{}^{\nu_1}}, \qquad \lambda_1 > 0, \nu_1 > 0, t_1 > 0 \text{ and} \tag{26}$$

$$f(t_2) = \lambda_2.\nu_2.t^{\nu_2-1}.e^{-\lambda_2.t_2{}^{\nu_2}}, \qquad \lambda_2 > 0, \nu_2 > 0, t_2 > 0 \tag{27}$$

The Expected values of 'T_1' and 'T_2' are

$$E(T_1) = \frac{1}{\lambda_1}.\Gamma\left(1+\frac{1}{\nu_1}\right) \text{ and} \tag{28}$$

$$E(T_2) = \frac{1}{\lambda_2}.\Gamma\left(1+\frac{1}{\nu_2}\right) \tag{29}$$

Substituting these values in the equation (18), we get the expected cost per a unit time as

$$Z = \{C_0\left(\frac{c.\theta}{c-1}\right) + C_1\{\left(S+\frac{1}{(1-\beta)}\right).h - \left(\frac{c.\theta}{c-1}\right) + n.E + \frac{\delta_1}{\lambda_1}.\Gamma(1+\frac{1}{\nu_1}) + \frac{\delta_2}{\lambda_2}.\Gamma(1+\frac{1}{\nu_2})\} + C_2.S.\alpha + \frac{(a+b.n)}{h}.\{\left(S+\frac{1}{(1-\beta)}\right).h + n.E + \frac{\delta_1}{\lambda_1}.\Gamma\left(1+\frac{1}{\nu_1}\right) + \frac{\delta_2}{\lambda_2}.\Gamma\left(1+\frac{1}{\nu_2}\right)\} + W\} / \{(1-\delta_1).T_0.S.\alpha + (S+\frac{1}{(1-\beta)}).h + n.E + \frac{\Gamma\left(1+\frac{1}{\nu_1}\right)}{\lambda_1} + \frac{\Gamma\left(1+\frac{1}{\nu_2}\right)}{\lambda_2}\} \tag{30}$$

Where, 'α' and 'β' are as defined in Equations (5) and (6) respectively.

for obtaining the optimal design parameters of the $\overline{X}$ chart, we differentiate 'Z' with respect to 'h' and 'n' and equate them to zero.

$\frac{\partial z}{\partial h} = 0$ implies,

$$\{\{(1-\delta_1).T_0.S.\alpha + \left(S+\frac{1}{(1-\beta)}\right).h + n.E + \frac{\Gamma\left(1+\frac{1}{\nu_1}\right)}{\lambda_1} + \frac{\Gamma\left(1+\frac{1}{\nu_2}\right)}{\lambda_2}\}.\{C_1.\left[S+\frac{1}{(1-\beta)}+h.R\right] + C_2.\alpha.R + \frac{(a+b.n)}{h}.\{S+\frac{1}{(1-\beta)}+h.R\} - \frac{(a+b.n)}{h^2}.\{\left(S+\frac{1}{(1-\beta)}\right).h + n.E + \frac{\delta_1}{\lambda_1}.\Gamma\left(1+\frac{1}{\nu_1}\right) + \frac{\delta_2}{\lambda_2}.\Gamma\left(1+\frac{1}{\nu_2}\right)\}\} - \{C_0\left(\frac{c.\theta}{c-1}\right) + C_1\{\left(S+\frac{1}{(1-\beta)}\right).h - \left(\frac{c.\theta}{c-1}\right) + n.E + \frac{\delta_1}{\lambda_1}.\Gamma(1+\frac{1}{\nu_1}) + \frac{\delta_2}{\lambda_2}.\Gamma(1+\frac{1}{\nu_2})\} + C_2.S.\alpha + \frac{(a+b.n)}{h}.\{\left(S+\frac{1}{(1-\beta)}\right).h + n.E + \frac{\delta_1}{\lambda_1}.\Gamma\left(1+\frac{1}{\nu_1}\right) + \frac{\delta_2}{\lambda_2}.\Gamma\left(1+\frac{1}{\nu_2}\right)\} + W\}.\{S + \frac{1}{(1-\beta)} + h.R + (1-\delta_1).T_0.S.\alpha.R\}\} / \{(1-\delta_1).T_0.S.\alpha + \left(S+\frac{1}{(1-\beta)}\right).h + n.E + \frac{\Gamma\left(1+\frac{1}{\nu_1}\right)}{\lambda_1} + \frac{\Gamma\left(1+\frac{1}{\nu_2}\right)}{\lambda_2}\}^2 = 0 \tag{31}$$

Where, 'α' and 'β' are as defined in Equations (5) and (6) respectively.

$\frac{\partial z}{\partial n} = 0$ implies,

$$\{\{(1-\delta_1).T_0.S.\alpha + \left(S+\frac{1}{(1-\beta)}\right).h + n.E + \frac{\Gamma\left(1+\frac{1}{\nu_1}\right)}{\lambda_1} + \frac{\Gamma\left(1+\frac{1}{\nu_2}\right)}{\lambda_2}\}.\{C_1.E + \frac{(a+b.n)}{h}.E + \left(\frac{b}{h}\right).\left[\left(S+\frac{1}{(1-\beta)}\right).h + n.E + \frac{\delta_1}{\lambda_1}.\Gamma\left(1+\frac{1}{\nu_1}\right) + \frac{\delta_2}{\lambda_2}.\Gamma\left(1+\frac{1}{\nu_2}\right)\right]\} - \{C_0\left(\frac{c.\theta}{c-1}\right) + C_1\{\left(S+\frac{1}{(1-\beta)}\right).h - \left(\frac{c.\theta}{c-1}\right) + n.E + \frac{\delta_1}{\lambda_1}.\Gamma(1+\frac{1}{\nu_1}) + \frac{\delta_2}{\lambda_2}.\Gamma(1+\frac{1}{\nu_2})\} + C_2.S.\alpha + \frac{(a+b.n)}{h}.\{\left(S+\frac{1}{(1-\beta)}\right).h + n.E + \frac{\delta_1}{\lambda_1}.\Gamma(1+\frac{1}{\nu_1}) + \frac{\delta_2}{\lambda_2}.\Gamma(1+\frac{1}{\nu_2})\} + W\}.E\} / \{(1-\delta_1).T_0.S.\alpha + \left(S+\frac{1}{(1-\beta)}\right).h + n.E + \frac{\Gamma\left(1+\frac{1}{\nu_1}\right)}{\lambda_1} + \frac{\Gamma\left(1+\frac{1}{\nu_2}\right)}{\lambda_2}\}^2 = 0 \tag{32}$$

Where, 'α' and 'β' are as defined in Equations (5) and (6) respectively.

Solving the equations (31) and (32) simultaneously for 'h' and 'n' using numerical techniques, we obtain the optimal time interval between successive samples (h^*) and the optimal sample size (n^*).

To study the effect of the random nature of 'T_1' and 'T_2' on the optimal design parameters we carry out the sensitivity analysis for the parameters 'λ_1', 'ν_1', 'λ_2' and 'ν_2' with the initial values of the other parameters as

c =2, θ = 5, $\delta_1 = \delta_2 = 1$, E= 0.01, a = 3, b = 0.01, T_0=1, C_2=5,W = 90,C_0=10, C_1=20, ξ=0.05, η=2, γ=2, δ=1, μ=2.5, σ=1, k=3, Δ=0.5 and are shown in Table-3.

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Table 3. Optimal values of n, h and Z for various values of λ_1, ν_1, λ_2 and ν_2

λ_1	ν_1	λ_2	ν_2	h*	n*	Z*
2	0.7	1	0.5	11.525	6	19.893
4	0.7	1	0.5	11.398	8	19.892
6	0.7	1	0.5	11.357	9	19.892
8	0.7	1	0.5	11.336	10	19.892
3	0.4	1	0.5	11.716	2	19.894
3	0.5	1	0.5	11.538	5	19.893
3	0.6	1	0.5	11.472	7	19.893
3	0.9	1	0.5	11.412	8	19.892
3	0.7	1.1	0.5	11.368	9	19.892
3	0.7	1.2	0.5	11.308	11	19.892
3	0.7	1.3	0.5	11.257	12	19.892
3	0.7	1.4	0.5	11.214	13	19.892
3	0.7	1	0.50	11.44	8	19.893
3	0.7	1	0.54	11.342	10	19.892
3	0.7	1	0.58	11.272	11	19.892
3	0.7	1	0.62	11.220	13	19.892

From Table 3, we observe that as the parameter 'λ_1' increases, the optimal values of 'n' is increasing, the optimal values of 'h' is decreasing and the expected cost per hour decreases for fixed values of the other parameters. When the parameter 'ν_1' increases , the optimal values of 'n' is increasing, the optimal values of 'h' is decreasing and the expected cost per hour decreases for fixed values of the other parameters. With respect to the parameter 'λ_2' as it increases , the optimal values of 'n' is increasing, the optimal values of 'h' is decreasing and the expected cost per hour remains constant for fixed values of the other parameters. As the parameter 'ν_2' increases , the optimal values of 'n' is increasing, the optimal values of 'h' is decreasing and the expected cost per hour decreases for fixed values of the other parameters. The optimal design parameters, n^* and h^* and the expected cost per a unit time are highly sensitive to the Weibull model parameters and by suitably estimating the model parameters we can have more accuracy in reducing the nonconformities and minimize the cost of Quality improvement program.

5. Conclusions:

In this paper we have proposed a Statistical economic design of $\bar{X}$ control chart for the variables having Johnson distribution as $\bar{X}$ distribution and the process in-control times follow a Pareto distribution. The Pareto distribution considered in this study can be applied for the processes which will run for a minimum period of time without non conformities. This distribution also includes the increasing and decreasing rates of failure. Minimizing the expected cost per a unit time, the optimal design parameters namely, sample size and time interval between successive samples are determined.

The numerical values indicate that the effect of Johnson distribution parameters and Pareto distribution parameters have significant effect on optimal design parameters. Another variation in this model is also considered by introducing randomness for 'T_1' and 'T_2' (Time to identify the assignable cause and time to repair). As a result of this modification it is observed that the model parameters of 'T_1' and 'T_2'will also significantly influence the design parameters. Sensitivity analysis carried out indicates that the optimal design parameters and the cost per a unit time are more sensitive towards the cost parameters than the other parameters. This design is much useful in quality control programs of production industries like chemicals, paints, films, etc.

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